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CONTROL OF LINEAR UNSTABLE SYSTEMS VIA THE DIRECT METHOD OF LIAPUNOV

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November 1967

This work has been sponsored by the
National Aeronautics and Space Administration
Research Grant NsG-309

TABLE OF CONTENTS

Chapter I	Introduction	1
Chapter II	Formulation of the Synthesis Technique	4
Chapter III	Application of the Technique	17
Chapter IV	The Regulator Problem	36
Chapter V	A Sixth Order Example	43
Chapter VI	Possible Areas for Extension	58

LIST OF ILLUSTRATIONS

		Page
Figure 2.1	Basic Configuration	6
Figure 2.2	Model Configuration	9
Figure 2.3	Non-Ideal Sign Function	13
Figure 2.4	Possible Imperfections	13
Figure 3.1	Possible Model Configuration	23
Figure 3.2	System with Model Similar to Plant	25
Figure 3.3	Second Order Example	31
Figure 3.4	Second Order Example - Phase Plane	34
Figure 3.5	Second Order Example - Error Response	35
Figure 4.1	Regulator Problem - Phase Plane	39
Figure 4.2	Regulator Problem - Actual Recoverable Set	40
Figure 4.3	Regulator Problem - Maximum R_w	41
Figure 5.1	The Continuous Form of the Mechanical Model	44
Figure 5.2	The Discrete Model of the Flexible Beam	45
Figure 5.3	Sixth Order Example	47
Figure 5.4	Sixth Order Model	53
Figure 5.5	Sixth Order System - Step Response	55
Figure 5.6	Sixth Order System - Response to an Initial Error	56

CHAPTER I

INTRODUCTION

In recent years there has been increasing interest in the application of Liapunov's direct method to the design of practical control systems.[1,2] The technique involves the definition of a non-linear controller which will cause the plant to track a phase-variable model in some region of the state space. The effect of transducer noise on the design technique has been examined by Taylor[3] and Jorgenson,[4] and their work represents a significant advance in the practical application of the design technique.

The technique is of particular interest when applied to systems which exhibit unstable open-loop response, since a region in the state space can be defined, within which stability can be assured. Such problems are commonplace in the chemical industry. The attitude control of a flexible missle also falls within this category, and in fact, it was the examination of the flexible missle problem which initiated the work on this thesis.

Most of the theory developed requires that the plant output be the lowest order phase variable of the plant. In theory, there is no reason why a plant, whose output is formed by a linear combination of the lowest order phase-variable and other plant states, cannot be transformed to the required canonic form by an appropriate transformation. This transformation has been investigated by various authors[5,6,7] for linear systems, and has been shown to exist when the conditions for controllability are satisfied.

The problem investigated in this thesis is the control of an unstable linear system whose output is directly related to the lowest order phase variable of the system. A bounded disturbance vector is included in the system equations.

In Chapter II the basic synthesis technique is developed. The approach is to first define a transformation to the required canonic form. A semi-definite Liapunov function is defined as suggested by Taylor,[3] and the synthesis procedure then follows the basic lines outlined by Lindorff.[1] The results of Taylor's work on transducer noise[3] is included along with the effect of a disturbance vector.

In Chapter III, the practical problems occuring in the application of the technique are investigated. The region in the state space in which the controller can insure stability is examined closely. It is noted that validity of the control law within a region is not sufficient to insure stability in that region. Another region is defined within which the control law is valid and system motion is constrained to lie within the region. Conservative estimates of these regions are noted as being useful in the actual design of the system.

The problem of selecting an appropriate switching line is examined in some detail, and it is noted that the semi-definite Liapunov function resulted in a much more practical constraint on the coefficients of the switching function than that developed by Monopoli.[1] Chapter III is concluded with an example illustrating the application of the design technique.

An interesting special case of the material of Chapters II and III is treated in Chapter IV. The problem comes about with the elimination of the model.

In Chapter V the synthesis technique is applied to a complex sixth order system. The system consists of a two-segment inverted pendulum, hinged in the center by a spring representing elastic stiffness, mounted on a frictionless cart. Control is exercised through a force acting horizontally on the cart. Changes in cart position are commanded by a reference input to the model. The results of simulation studies of the system are included in this chapter.

CHAPTER II

FORMULATION OF THE SYNTHESIS TECHNIQUE

Introduction

The problem originally investigated by the author was the control of an inverted pendulum mounted on a frictionless cart. An attempt to apply the synthesis technique as outlined by Lindorff[2] was unsuccessful for three reasons: 1) The plant output (cart position) was not the lowest order phase variable of the system as required. 2) The selection of the switching line by the techniques of Monopoli[1] was judged to be too cumbersome to be of practical use in the design of higher order systems. 3) The effect of disturbances was not treated in sufficient detail to aid in the design of the system.

The first limitation was eliminated by simply transforming the equations to the required form. It should be noted that the plant output must be defined such that the linear combination of states forming the output includes the lowest order phase-variable. This requirement is necessary to insure that the reference input corresponds to a stable equilibrium point.

The cumbersome equation of Monopoli[1] used for the selection of the switching line was avoided when the semi-definite Liapunov function suggested by Taylor was employed.

The effect of disturbances in the system was included in a manner similar to Taylor's treatment of transducer noise.[3]

Statement of the Problem

The system is assumed to be described by an equation of the form

$$\dot{\underline{y}} = A\underline{y} + \underline{f}u + \underline{g}(\underline{y},\underline{z},t), \tag{2.1}$$

where

$\underline{y}$ - n dimensional state vector

u - controlled force

$\underline{z}$ - q dimensional disturbance vector $(q < n)$

A - n x n constant matrix

$\underline{f}$ - n x 1 constant vector

$\underline{g}$ - n dimensional disturbance function

The system described by equation 2.1 is assumed to be classically unstable but controllable. The disturbance $\underline{g}(\underline{y},\underline{z},t)$ and the controlled force u are bounded.

The objective is to define a controller which will guarantee stability within some region of the state space in the presence of the disturbance vector, $\underline{g}(\underline{y},\underline{z},t)$, and to provide an input which will allow the system to be commanded to move in some prescribed manner within this region.

The approach (refer to Figure 2.1) to be taken will be to first define a transformation on the system equations to convert them to a canonic form. A phase variable model with a reference input will be chosen, such that the plant will be able to track the model in some region of the state space. The controller and limits on certain of the model states will be defined by first choosing a Liapunov function, v, which is positive-semidefinite in the error space defined by the

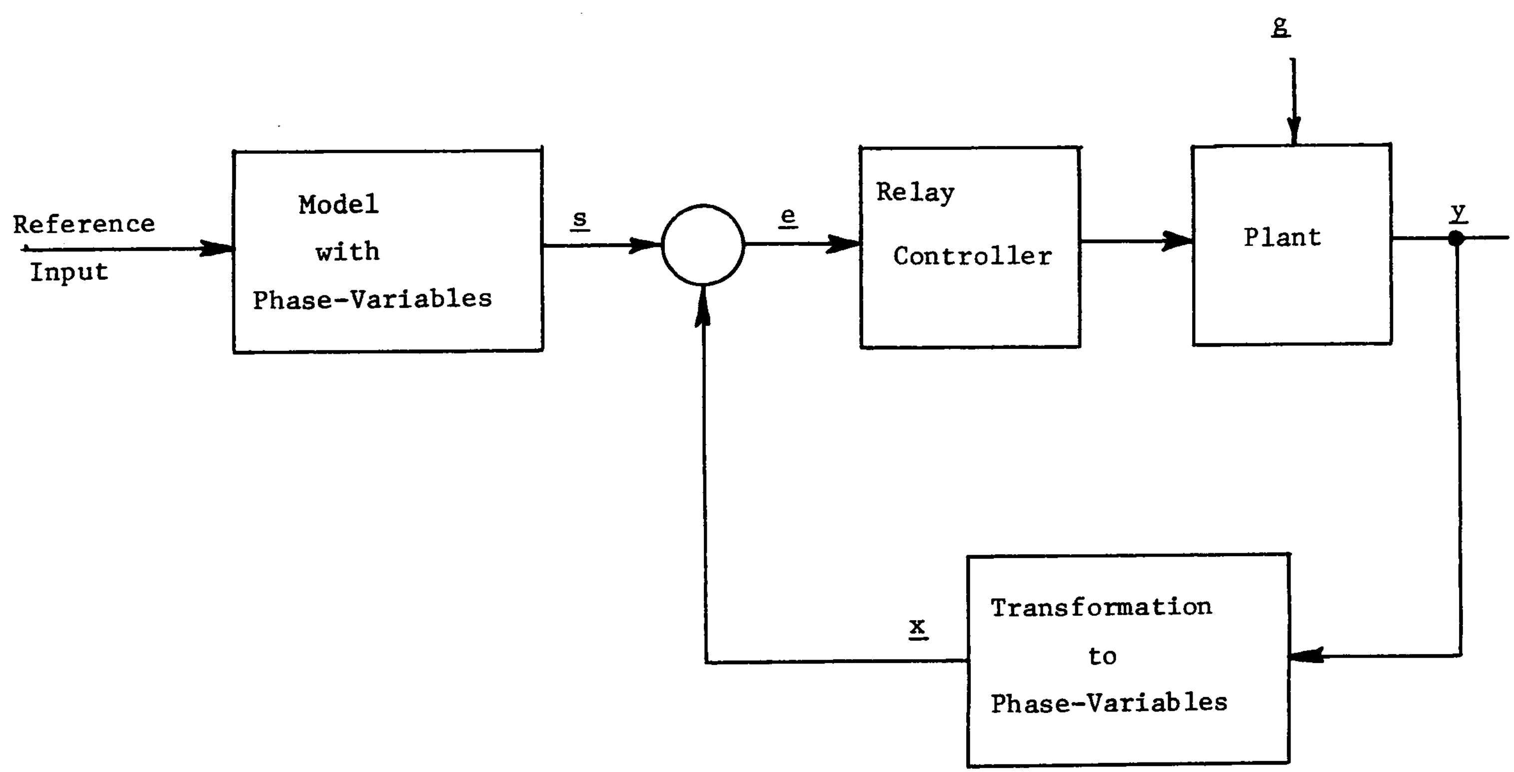

Figure 2.1 BASIC CONFIGURATION

difference between the transformed plant states and the model states. A relay controller will then be chosen such that 1) $\dot{v}$ is negative semi-definite in some region of the state space, where $\dot{v} = 0$ only when $v = 0$, and 2) Motion near $v = 0$ is stable and bounded even in the presence of an imperfection in the switching action of the relay, and noise on certain of the plant states.

Transformation to Canonic Form

It is desired to find a transformation

$$\underline{y} = K\underline{x}, \tag{2.2}$$

which transforms equation 2.1 to the canonic form

$$\dot{\underline{x}} = A_0\underline{x} + \underline{f}_0 u + \underline{h}(\underline{x},\underline{z},t), \tag{2.3}$$

where the forms of A_0 and $\underline{f}_0$ are

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & \cdots\cdots & 0 \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\ 0 & 0 & 0 & \cdots\cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots\cdots & -a_n \end{bmatrix} \tag{2.4}$$

and

$$\underline{f}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{2.5}$$

The existence of the matrix K is guaranteed if the system is controllable. The computation of the matrix K is straight-forward and has been examined by Rane[5] and others.[6,7]

Formulation of the Model

The model is defined of the form shown in Figure 2.2 such that the variables in the model space $\underline{s}$ are phase-variable, i.e.,

$$\begin{aligned} \dot{s}_1 &= s_2 \\ \dot{s}_2 &= s_3 \\ &\vdots \\ \dot{s}_{n-1} &= s_n, \end{aligned} \tag{2.6}$$

and the nature of $\dot{s}_n$ will be determined by the stability requirements to be developed.

An error space may be defined as in Figure 2.1,

$$\underline{e} = \underline{s} - \underline{x}. \tag{2,7}$$

In order that the plant be guaranteed to track the model, it must be shown that the system is asymptotically stable in a region of the error space.

The Liapunov Function

Consider the Liapunov function given by

$$v = \tfrac{1}{2}\underline{e}^T P \underline{e}, \tag{2.8}$$

where P is chosen to be positive semi-definite symmetric matrix of rank one. The P matrix is then of the form

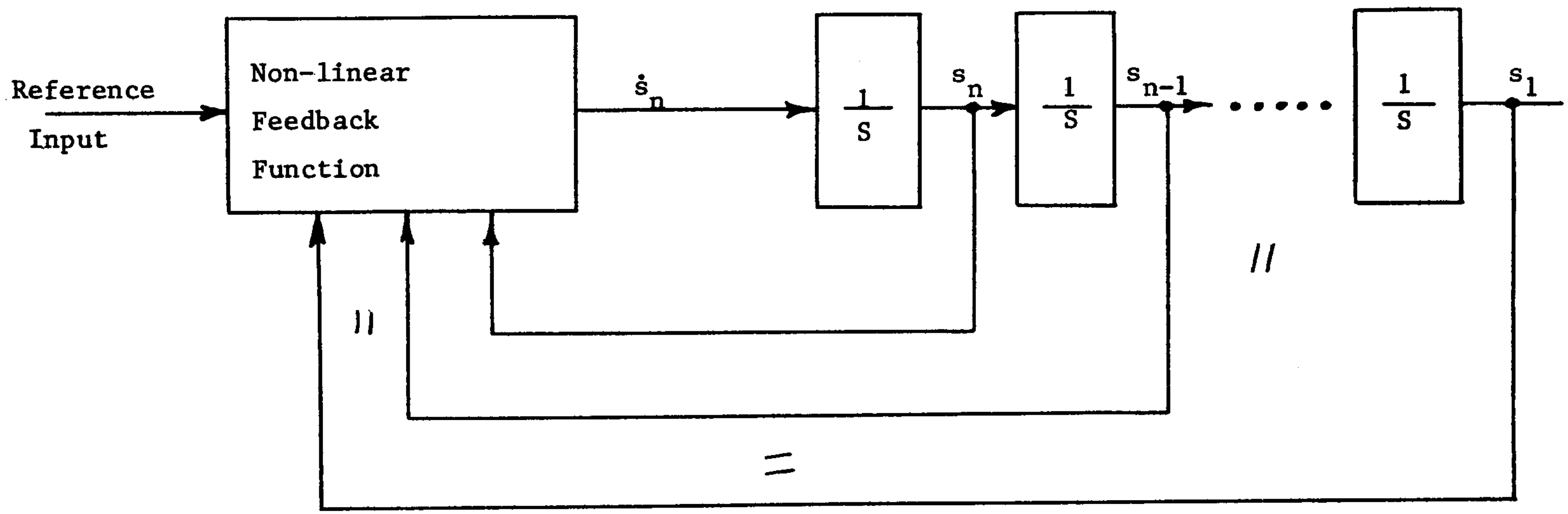

Figure 2.2 MODEL CONFIGURATION

$$P = \begin{bmatrix} \frac{P_{1n}^2}{P_{nn}} & \frac{P_{1n}P_{2n}}{P_{nn}} & \cdots\cdots & P_{1n} \\ \frac{P_{1n}P_{2n}}{P_{nn}} & \frac{P_{2n}^2}{P_{nn}} & \cdots\cdots & P_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P_{1n} & P_{2n} & \cdots\cdots & P_{nn} \end{bmatrix} \tag{2.9}$$

and v can be expressed as

$$v = \frac{1}{2P_{nn}} (P_{1n}e_1 + P_{2n}e_2 + \ldots. + P_{nn}e_n)^2. \tag{2.10}$$

Defining

$$\gamma = P_{1n}e_1 + P_{2n}e_2 + \ldots + P_{nn}e_n, \tag{2.11}$$

allows equation 2.10 to be written as

$$v = \frac{1}{2P_{nn}} \gamma^2,$$

which is positive semidefinite if

$$P_{nn} > 0. \tag{2.12}$$

The time derivative of v can be expressed as

$$\dot{v} = \frac{1}{P_{nn}} \gamma\dot{\gamma}, \tag{2.13}$$

or

$$\dot{v} = \frac{1}{P_{nn}} \gamma (P_{1n}\dot{e}_1 + P_{2n}\dot{e}_2 + \ldots + P_{nn}\dot{e}_n). \tag{2.14}$$

From equation 2.7 it can be observed that

$$\underline{\dot{e}} = \underline{\dot{s}} - \underline{\dot{x}} \tag{2.15}$$

Substituting for $\underline{\dot{s}}$ and $\underline{\dot{x}}$ gives

$$
\begin{aligned}
\dot{e}_1 &= e_2 - h_1 \\
\dot{e}_2 &= e_3 - h_2 \\
&\vdots \\
\dot{e}_{n-1} &= e_n - h_{n-1} \\
\dot{e}_n &= \dot{s}_n - u + a_1 x_1 + \ldots + a_n x_n - h_n .
\end{aligned}
\tag{2.16}
$$

Employing equation 2.16 and rearranging terms allows $\dot{v}$ to be written as

$$
\begin{aligned}
\dot{v} = -\gamma(u &- \dot{s}_n - a_1 x_1 - a_2 x_2 \ldots - a_n x_n + \frac{P_{1n}}{P_{nn}} h_1 + \ldots \\
&\ldots + \frac{P_{n-1n}}{P_{nn}} h_{n-1} + h_n - \frac{P_{1n}}{P_{nn}} e_2 - \ldots - \frac{P_{n-1n}}{P_{nn}} e_n).
\end{aligned}
\tag{2.17}
$$

<u>The Control Law</u>

It is desired to define a control law

$$u = f(\underline{e}) \tag{2.18}$$

such that: 1) $\dot{v} < 0$ when $v \neq 0$, and 2) Motion is stable near the surface $v = 0$. It is hypothesized that the desired u is of the form

$$u = U_0 \mathrm{SGN}(\gamma), \tag{2.19}$$

where U_0 is made large enough to control the sign of the expression in parenthesis in equation 2.17. More precisely, a region R in the state space, within which $\dot{v} < 0$ when $v \neq 0$, is defined by

$$
\begin{aligned}
\mathrm{SGN}(\gamma) = \mathrm{SGN}(U_0 \mathrm{SGN}(\gamma) &- \dot{s}_n - a_1 x_1 - a_2 x_2 \ldots\ldots \\
&- a_n x_n + \frac{P_{1n}}{P_{nn}} h_1 + \frac{P_{2n}}{P_{nn}} h_2 \ldots\ldots\ldots + \\
&\frac{P_{n-1n}}{P_{nn}} h_{n-1} + h_n - \frac{P_{1n}}{P_{nn}} e_2 - \frac{P_{2n}}{P_{nn}} e_3 \ldots - \frac{P_{n-1n}}{P_{nn}} e_n).
\end{aligned}
\tag{2.20}
$$

The Non-ideal Controller

At this point it will be realized that in practical systems γ cannot be measured exactly, and the u actually implemented will be of the form

$$u = U_0 \mathrm{SGN}(\gamma'), \tag{2.21}$$

where

$$\begin{aligned} \gamma' &= \gamma + \gamma_N \\ &= \gamma + P_{1n}N_1 + \ldots + P_{nn}N_n, \end{aligned} \tag{2.22}$$

where N_i represents the measurement noise on the variable x_i. Furthermore, it will be assumed that there is also an imperfection in the relay switching such that for

$$|\gamma| < \delta_R \tag{2.23}$$

u is not uniquely defined. Thus the SGN function is implemented as shown in Figure 2.3. Two possible forms of the imperfection are illustrated in Figure 2.4.

The noise N_i on each measurement is assumed to be bounded, so that there exists some maximum value for $|\gamma_N|$, which will be defined as δ_N, thus

$$\delta_N = \mathrm{MAX}(|\gamma_N|). \tag{2.24}$$

δ_N can be visualized as an additional region for which the relay output is not uniquely defined, such that the effective region for which u is undefined is given by

$$|\gamma| < \delta_T, \tag{2.25}$$

where

$$\delta_T = \delta_R + \delta_N. \tag{2.26}$$

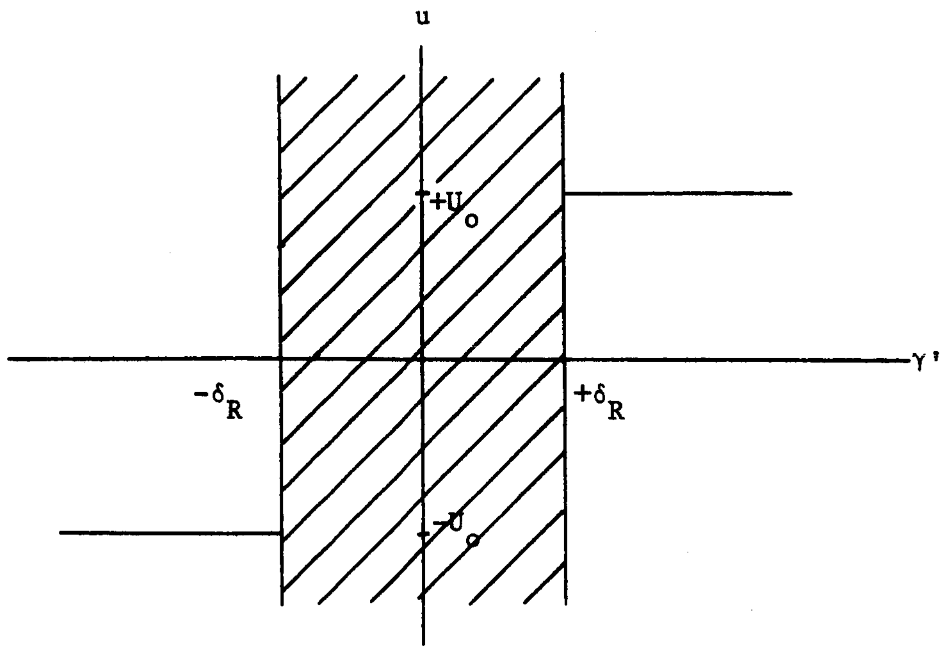

Figure 2.3 NON-IDEAL SIGN FUNCTION

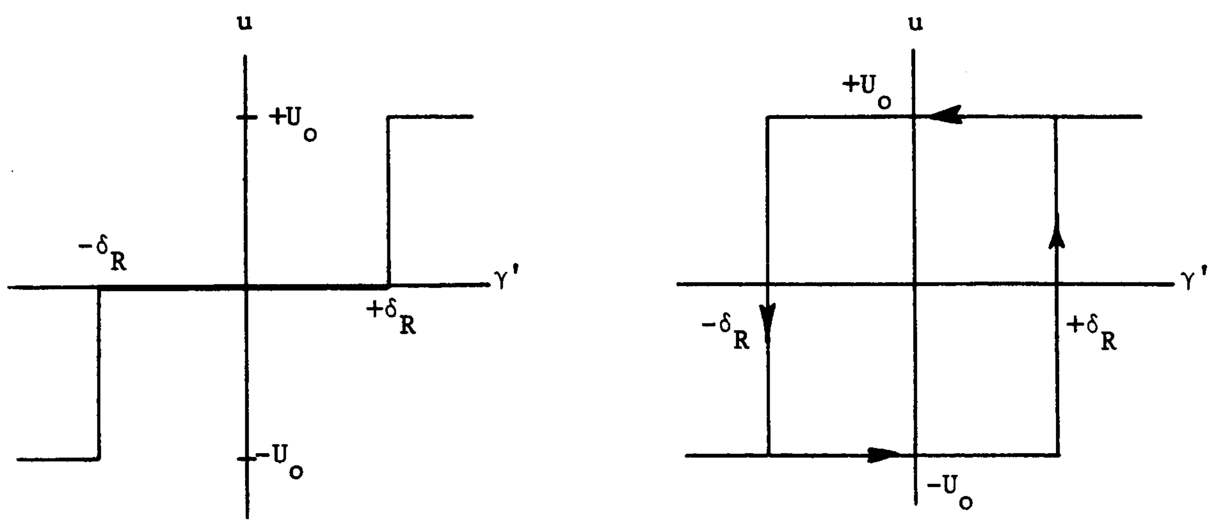

Figure 2.4 POSSIBLE IMPERFECTIONS

Thus, the action of the non-ideal relay can be described by

$$\begin{array}{ll} u = +U_0 & \gamma > \delta_T \\ 0 < |u| < |U_0| & |\gamma| < \delta_T \\ u = -U_0 & \gamma < \delta_T. \end{array} \tag{2.27}$$

Stability of the System

Stability of the system of equation 2.1 with the non-ideal controller of equation 2.27 will be investigated in two steps. First, it will be shown when $|\gamma| < \delta_T$ that state motion asymptotically approaches the hyperplane $\gamma = 0$, and secondly, within the hyperstrip $|\gamma| < \delta_T$ that state motion is stable and bounded.

Motion When $|\gamma| > \delta_T$

In the region within R for which $|\gamma| > \delta_T$, the output of the non-ideal relay will be identical to the ideal u

$$u = U_0 \mathrm{SGN}(\gamma) \tag{2.19}$$

and thus $\dot{v} < 0$, and the state vector will approach the hyperplane $\gamma = 0$ asymptotically.

Motion Within the Hyperstrip $|\gamma| < \delta_T$

The technique of Taylor[3] will be used to show that the motion within the hyperstrip $|\gamma| < \delta_T$ is bounded and stable. Motion in the error space is given by equation 2.14

$$\underline{\dot{e}} = \begin{bmatrix} e_2 - h_1 \\ e_3 - h_2 \\ \vdots \\ e_n - h_{n-1} \\ \dot{s}_n - u + a_1 x_1 + \ldots + a_n x_n - h_n \end{bmatrix} \tag{2.14}$$

Within the hyperstrip u is given by

$$0 \leq |u| \leq U_0, \tag{2.28}$$

and γ is known to be bounded,

$$\gamma = \beta \tag{2.29}$$

where

$$0 \leq |\beta| < \delta_T. \tag{2.30}$$

Solving equation 2.29 for e_n gives

$$e_n = \frac{1}{P_{nn}}(\beta - P_{1n}e_1 - P_{2n}e_2 - \dots - P_{n-1n}e_{n-1}) \tag{2.31}$$

Substituting this expression into equation 2.14 gives the reduced system.

$$\underline{\dot{e}}' = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & & 1 \\ -\frac{P_{1n}}{P_{nn}} & -\frac{P_{2n}}{P_{nn}} & -\frac{P_{3n}}{P_{nn}} & & -\frac{P_{n-1n}}{P_{nn}} \end{bmatrix} \underline{e}' + \begin{bmatrix} -h_1 \\ -h_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{\beta}{P_{nn}} \quad -h_{n-1} \end{bmatrix}$$

$$= C\underline{e}' + \underline{\alpha} \tag{2.32}$$

If $P_{1n}, P_{2n}, \dots, P_{n-1n}$, and P_{nn} are chosen such that the C matrix is a stability matrix, the motion in the error space is stable. Furthermore, since $\underline{h}$ and β are both bounded the input to this reduced system is bounded, and so, the output is bounded.

Thus the ultimate bound of Taylor including a disturbance vector is simply the boundary of the reachable set of the system described by equations 2.31 and 2.32.

The determination of the reachable set of a stable system with bounded input is analogous to finding the recoverable set of an unstable system with a bounded input. This problem has been investigated by Higdon[8] and Lemay[9]. The boundary could be readily found for the system without a disturbance (i.e. $\underline{h} = \underline{0}$). The determination of the bound when $\underline{h} \neq \underline{0}$ is theoretically possible, but the solution by existing techniques becomes quite formidable. For this reason, the actual technique of determining the reachable set will not be discussed in detail.

CHAPTER III

APPLICATION OF THE SYNTHESIS TECHNIQUE

Introduction

The application of the synthesis technique of Chapter II brings up many questions, since some of the equations in Chapter II, although theoretically meaningful, are of little direct use to the designer.

The region of operation is discussed in considerable detail, and a conservative approximation of this region is defined in the plant state space to aid the designer in the evaluation of a particular design. Some of the aspects of choosing a model are discussed briefly.

The selection of the switching line is examined in some detail. The expression defining the ultimate error bound including the effect of a disturbance is developed. The effect of the switching line on convergence time is also discussed, including a special case of chatter motion on the switching line.

Much of the material of Chapters II and III is illustrated with a second order example corresponding to the inverted pendulum controlled by a torque source at the pivot in the presence of a wind disturbance.

Controllable Zone

The region R in the state space for which the control law assures $\dot{v} < 0$ is given by equation 2.20. In this region the system can cause the plant to track the model with a bounded error. However, it is possible for system trajectories to leave R while obeying the control law, and, of course, as soon as the plant state leaves R the stability of the system can no longer be guaranteed.

This result is to be expected when one considers the freedom provided in the design of the model, and it is recognized that the system trajectories are determined by the model. In fact, the synthesis technique does not even require the model to be stable!

The controllable zone, R', will be defined as that region within R, consisting of all points A and B for which it is possible for the model to cause the system to move from A to B and from B to A without leaving the region R.

The regions R and R', although theoretically meaningful, are judged to be much too complex to be helpful in the actual design of a practical system, since R and R' are a function of $\underline{P}$, $\underline{h}$, $\underline{x}$, $\underline{s}$, $\dot{s}_n$, and $\underline{e}$.

Two much more meaningful regions would be the worst case regions in $\underline{x}$ space corresponding to the regions R and R'. To define these regions, a conservative subregion of R will be defined in $\underline{x}$ space with necessary assumptions, and a conservative controllable zone will be defined with respect to this region.

The region R is defined by equation 2.20

$$\begin{aligned} SGN(\gamma) = SGN(&U_0 SGN(\gamma) - \dot{s}_n - a_1x_1 - a_2x_2 - \dots \\ &\dots -a_nx_n + \frac{P_{1n}}{P_{nn}}h_1 + \frac{P_{2n}}{P_{nn}}h_2 + \dots + \frac{P_{n-1n}}{P_{nn}}h_{n-1} + \\ &+h_n - \frac{P_{1n}}{P_{nn}}e_2 - \frac{P_{2n}}{P_{nn}}e_3 - \dots - \frac{P_{n-1n}}{P_{nn}}e_n). \end{aligned} \tag{2.20}$$

Equation 2.20 would certainly be satisfied if U_0 were chosen large enough to control the sign of the terms on the right hand side,

$$U_0 > \left| -\dot{s}_n - a_1x_1 - a_2x_2 - \ldots - a_nx_n + \frac{P_{1n}}{P_{nn}} h_1 + \ldots \right.$$

$$\ldots + \frac{P_{n-1n}}{P_{nn}} h_{n-1} + h_n - \frac{P_{1n}}{P_{nn}} e_2 - \frac{P_{2n}}{P_{nn}} e_3 - \ldots \tag{3.1}$$

$$\left. \ldots - \frac{P_{n-1n}}{P_{nn}} e_n \right| .$$

Since the elements of $\underline{h}$ are bounded, the worst case for the terms involving $\underline{h}$ in equation 3.1 is simply the maximum value of these terms, H,

$$H = \mathrm{MAX}\left(\left| \frac{P_{1n}}{P_{nn}} h_1 + \frac{P_{2n}}{P_{nn}} h_2 + \ldots + \frac{P_{n-1n}}{P_{nn}} h_{n-1} + h_n \right| \right). \tag{3.2}$$

It will be assumed that the model is implemented in such a way that a magnitude constraint is imposed on $\dot{s}_n$,

$$\left| \dot{s}_n \right| \leq M. \tag{3.3}$$

In order that the terms in $\underline{e}$ in equation 3.1 be bounded, it is necessary to require that the error motion be within Taylor's bound, and that the system motion remain within R. This assumption is not overly restrictive since the error could be initially set within the error bound. The worst case of the terms involving $\underline{e}$ is then given by

$$E = \mathrm{MAX}\left(\left| - \frac{P_{1n}}{P_{nn}} e_2 - \frac{P_{2n}}{P_{nn}} e_3 - \ldots - \frac{P_{n-1n}}{P_{nn}} e_n \right| \right). \tag{3.4}$$

R_w, the conservative region in $\underline{x}$ space, corresponding to R, can now be defined as that region satisfying

$$U_0 - M - E - H \geq \left| -a_1x_1 - a_2x_2 - \ldots - a_nx_n \right| , \tag{3.5}$$

where it is assumed that the error motion is within the error bound of Taylor.

The worst case controllable zone, R'_w, with respect to R_w will be defined as that region within R_w, consisting of all points A and B for which it is possible for the model to cause the plant to move from A to B and from B to A without leaving R_w.

The region R_w, although conservative, gives the designer some feeling for the region in which the control law will cause the system to exhibit stable motion.

Summarizing briefly, the region R is that region within which the plant can track the model with a bounded error. Outside R the plant cannot track the model with a bounded error. However, brief excursions outside R can occur for which the system will remain stable. It was noted that there were points in R for which it was impossible for the system to force the state trajectory to remain within R, and thus a second region R' was defined as that region contained in R, within which the system trajectories could be forced to remain within R. The regions R and R' were judged to be of limited use in an actual design, and two new regions R_w and R'_w were defined in $\underline{x}$ space representing a conservative estimate of R and R'. The derivation of R_w and R'_w required the assumption that the system error was always within the error bound.

Selection of the Model

The selection of the model surely represents one of the most important decisions facing the designer, since both the system response and the region of operation is determined largely by the model. A discussion of what form of model is best will be avoided in this paper, however, it is instructive to determine that model which maximizes R'_w with respect to a given region R_w in an ideal system. An ideal system will be defined as a system which has an error bound of essentially zero size.

Consider the model represented by the system

$$\begin{aligned} \dot{s}_1 &= s_2 \\ \dot{s}_2 &= s_3 \\ &\vdots \\ \dot{s}_{n-1} &= s_n \\ \dot{s}_n &= \eta \end{aligned} \tag{3.6}$$

where η represents a bounded input

$$|\eta| \leq M. \tag{3.7}$$

For a particular system, M could be determined by equation 3.5 and could possibly be a function of U_0 and $\underline{x}$. However, throughout this Chapter it will be assumed that the designer has fixed M at some constant value consistent with his specifications.

The controller for the model is completely in the hands of the designer. Suppose the controller was optimal with respect to some cost function J,

$$J = \int_{t_0}^{t} f(\underline{s},t)dt \tag{3.8}$$

and that a state space constraint was imposed, such that $\underline{x}$ must remain within R, where in this ideal case of zero error, $\underline{s} = \underline{x}$. The constraint is such that infinite cost would be associated with motion outside R. Pontryagin's maximum principle then determines a $\eta(t)$ which minimizes J. Certainly this model results in the largest R_w' for the given R_w, since if this were not true there would have to exist two points $\underline{s}_A$ and $\underline{s}_B$ for which the cost J was not minimized.

In most systems it would not be practical, and maybe not even desirable, to construct a model of the form above. Even so, the existence of this model could serve as a guideline in the design of a more practical model.

Another important aspect of the model is its input. For a given model there is a set of inputs, representing the permissable set, which will not cause system motion to leave R.

In a practical application it may be desirable to choose a model where the input is bounded. If the bound on the input were chosen such that the motion of the model within its reachable set could not cause $\underline{x}$ to leave R_w, then stability could be guaranteed for all inputs and all permissable disturbances. A possible form for such a model is illustrated in Figure 3.1.

Although a detailed discussion of the many considerations to be weighed in the designing of a model is not the purpose of this section, a few points deserve mentioning. In most systems, it would be assumed that initially the initial conditions of the model would be chosen such that the error $\underline{e}$ at time t_0 would be within Taylor's bound, so that, the system error would remain within this bound provided the system state remained within R. Thus it is implied that plant transients due to steps are determined totally by the model. Furthermore, the model could be designed to give specific system trajectories that may be difficult to obtain by conventional techniques.

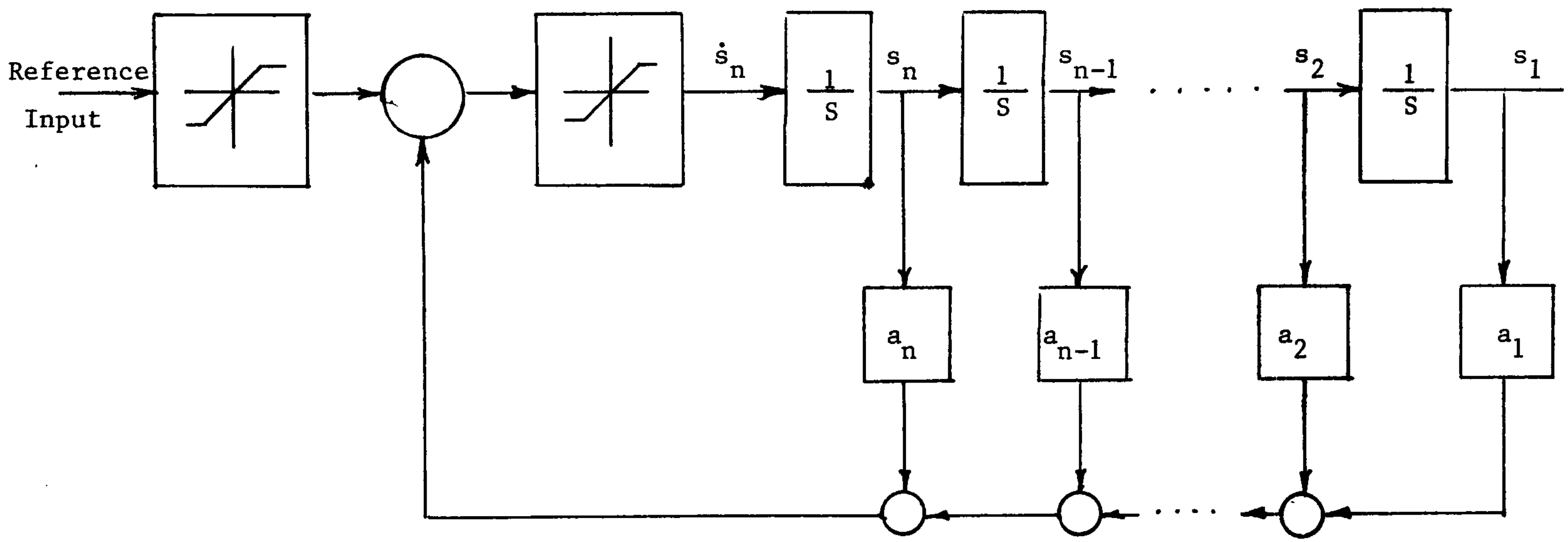

Figure 3.1 POSSIBLE MODEL CONFIGURATION

The problem of designing a model to system specifications poses another problem in so far as the model states do not correspond with the measured variables of the system. The existence of this problem suggests that it might be advantageous to treat a model whose form is similar to the plant. The basic form of this control configuration is shown in Figure 3.2. The vectors having the same basis as the original plant variables are denoted by a tilde. Vectors not so designated have a basis corresponding to the canonic space. For this model

$$\underline{\tilde{s}} = (K)\underline{s}, \tag{3.9}$$

$$\underline{\tilde{e}} = (K)\underline{e}, \tag{3.10}$$

and

$$\tilde{P} = \underline{P}^T(K) \tag{3.11}$$

Selection of the Switching Line

In addition to the designing of the model the designer must also choose a switching line

$$\gamma = 0. \tag{3.12}$$

For convenience a vector $\underline{P}$ is defined as

$$\underline{P} = \begin{bmatrix} P_{1n} \\ P_{2n} \\ \cdot \\ \cdot \\ \cdot \\ P_{nn} \end{bmatrix}, \tag{3.13}$$

and thus,

$$\gamma = \underline{P}^T\underline{x}. \tag{3.14}$$

When selecting γ, the designer need only satisfy the requirements that the matrix, C, in equation 2.32 be a stability matrix,

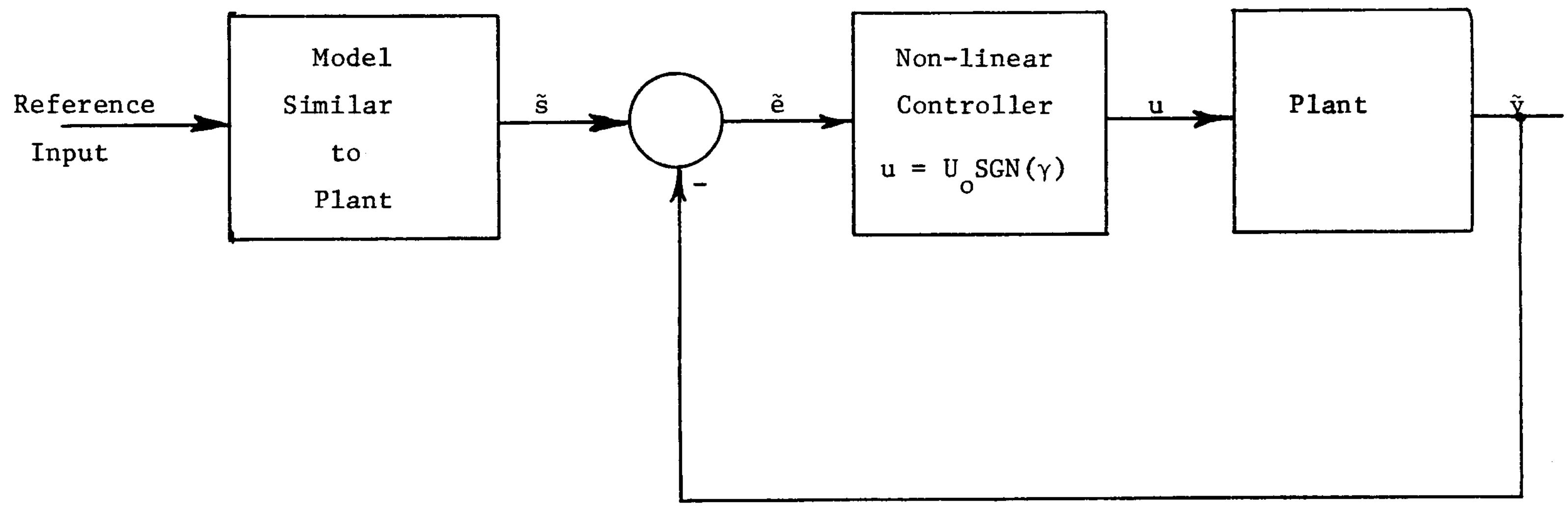

Figure 3.2 SYSTEM WITH MODEL SIMILAR TO PLANT

$$C = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & 1 \\ -\frac{P_{1n}}{P_{nn}} & -\frac{P_{2n}}{P_{nn}} & -\frac{P_{3n}}{P_{nn}} & \cdots & -\frac{P_{n-1n}}{P_{nn}} \end{bmatrix} \qquad (3.15)$$

and that P_{nn} be chosen such that

$$P_{nn} > 0. \qquad (3.16)$$

The constraint that these requirements place on $\underline{P}$ can be readily ascertained by the application of the Routh-Hurwitz technique to the characteristic equation of the matrix C, $\phi_C(\lambda)$

$$\phi_C(\lambda) = P_{nn}\lambda^{n-1} + P_{n-1n}\lambda^{n-2} + \ldots + P_2\lambda + P_1. \qquad (3.17)$$

The switching line affects three aspects of the systems response:

1) The region of operation

2) The error convergence

3) The ultimate error bound.

The effect of $\underline{P}$ on each of these aspects will be discussed in the following paragraphs.

<u>The Effect of $\underline{P}$ on the Region of Operation</u>

For the purposes of this section it will be assumed that the region of operation is represented by R_w'. The region R_w' varies directly with the region R_w, so for the purposes of a qualitative discussion of how $\underline{P}$ affects R_w', it will be sufficient to discuss how $\underline{P}$ affects R_w.

The region, R_w, is defined by equation 3.5,

$$U_0 - M - E - H \geq \left| -a_1x_1 - \ldots -a_nx_n \right| \qquad (3.5)$$

where it is assumed that the error is within the ultimate error bound. The only terms involving $\underline{P}$ are the terms E and H, where

$$E = \mathrm{MAX}\left(\left| - \frac{P_{1n}}{P_{nn}} e_2 - \ldots - \frac{P_{n-1n}}{P_{nn}} e_n \right|\right) \tag{3.4}$$

and

$$H = \mathrm{MAX}\left(\left| \frac{P_{1n}}{P_{nn}} h_1 + \frac{P_{2n}}{P_{nn}} h_2 + \ldots + \frac{P_{n-1n}}{P_{nn}} h_{n-1} + h_n \right|\right). \tag{3.2}$$

Thus, the switching line will determine the value of the quantity (E + H), and the size of the region R_w can be adjusted to some extent by the designer. In fact the region R_w could be maximized with respect to $\underline{P}$ by choosing $\underline{P}$ within its constraints to minimize the quantity (E + H). Jorgenson[4] used this approach to some extent in his work on the design of a physical system with noise.

The Effect of $\underline{P}$ on Error Convergence

In most systems, the error $\underline{e}$ will probably be initially set within the error bound of Taylor, and thus the error will be constrained to remain within this bound, assuming motion is never outside R. The problem of error convergence could arise, however, in systems where it is difficult or impossible to initially set the error within the error bound, or after a system has experienced a bounded excursion outside the region R. Although exact analysis of the error motion is not conceivable since the motion is a function of $\underline{h}$, $\underline{x}$, $\underline{s}$, $\dot{s}_n$, and $\underline{e}$, some special aspects of the motion may be treated. It will be assumed throughout this section that the system is operating within R.

For the sake of discussion the error trajectory can be treated as consisting of two parts. The first part is the section of the trajectory outside the strip $|\gamma| \leq \delta_T$, where the controller forces the error motion to approach the line $\gamma = 0$ at all times by the application of a constant force (either $+U_0$ or $-U_0$). The second part consists of the motion, once the strip has been reached, where the controller forces the system to remain within the strip. The motion outside the strip is the most difficult to analyze, and little can be said other than the motion approaches $\gamma = 0$ asymptotically. The motion within the strip is somewhat more tractable and the rest of this section will be devoted to the analysis of the motion within the strip.

The motion within the strip was treated previously, and the equation describing the motion was found to be

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{e}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & 1 \\ -\frac{P_{1n}}{P_{nn}} & -\frac{P_{2n}}{P_{nn}} & -\frac{P_{3n}}{P_{nn}} & \cdots & -\frac{P_{n-1n}}{P_{nn}} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_{n-1} \end{bmatrix} + \begin{bmatrix} -h_1 \\ -h_2 \\ \cdot \\ \cdot \\ \cdot \\ \frac{\beta}{P_{nn}} - h_{n-1} \end{bmatrix}$$

$$= C\underline{e}' + \underline{\alpha}, \tag{2.32}$$

and

$$e_n = \frac{1}{P_{nn}}(\beta - P_{1n}e_1 - P_{2n}e_2 - \ldots - P_{n-1}e_{n-1}). \tag{2.33}$$

The homogeneous response of this system is dependent only on $\underline{P}$, while $\underline{\alpha}$ is a function of P_{nn}, β and $\underline{h}$.

It is helpful at this point to present the analysis of a special case of the motion within the strip.

Special Case - The Ideal System

In an ideal system with no noise or disturbances, and $\delta_T = 0$, the $|\gamma| \leq \delta_T$ reduces to a line. This special case is simply a limiting case of equation 2.32 with $\underline{h} \to 0$ and $\delta_T \to 0$, thus the error motion along the line $\gamma = 0$ for the ideal system is given by

$$\dot{\underline{e}}' \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & & 0 & & 1 \\ -\frac{P_{1n}}{P_{nn}} & \cdots & \cdots & \cdots & \cdots & -\frac{P_{n-2n}}{P_{nn}} & & -\frac{P_{n-1n}}{P_{nn}} \end{bmatrix} \underline{e}', \tag{3.18}$$

and is determined completely by the $\underline{P}$ vector.

The Effect of $\underline{P}$ on the Ultimate Error Bound

As shown in section II, the ultimate error bound is simply the reachable set of the system described by equation 2.32.

$$\dot{\underline{e}}' = C\underline{e}' + \underline{\alpha}, \tag{2.32}$$

and thus, the bound is a function of $\underline{P}$, $\underline{h}$ and δ_T.

It should be noted that choosing $\underline{P}$ to minimize the effective imperfection in the relay due to noise, δ_N, will not necessarily result in the smallest error bound, since the bound is a function of both $\underline{P}$ and $\underline{\delta}_T$. It does follow that reducing δ_R will reduce the error bound.

Illustrative Example

At this point it is instructive to provide a simple example to illustrate the synthesis technique. A second order example is chosen so that the various regions can be readily displayed. The example is also directly related to the system to be studied in Chapter V.

The example chosen, refering to Figure 3.3 consists of an inverted pendulum controlled by a torque source u' acting at the pivot. The disturbance, corresponding to wind, acts on the mass m in a horizontal direction. The mass m is assumed to appear as a point mass on the end of a massless rod of length l. The equation of motion of this system is

$$\ddot{\phi} - \frac{g}{l}\sin\phi = \frac{u'}{ml^2} - \frac{\cos\phi}{ml}d'. \tag{3.19}$$

Linearizing by letting $\sin\phi = \phi$ and $\cos\phi = 1$ and defining

$$\lambda^2 = \frac{g}{l}, \tag{3.20}$$

$$u = \left(\frac{1}{ml^2}\right)u', \tag{3.21}$$

$$d = \frac{1}{ml}d'. \tag{3.22}$$

gives the linearized equation of motion as

$$\ddot{\phi} - \lambda^2\phi = u - d. \tag{3.23}$$

Defining new variables $x_1 = \phi$ and $x_2 = \dot{\phi}$ enables the equation of motion to be expressed in matrix form as

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ d \end{bmatrix}. \tag{3.24}$$

Since this equation is already in canonic (phase-variable) form no transformation is required.

The control law for this system is

$$u = U_0 SGN(P_{12}x_1 + P_{22}x_2). \tag{3.25}$$

The region R in which $\dot{v} < 0$ is given by 2.18

$$SGN(\gamma) = SGN(U_0 SGN(\gamma) - \dot{s}_n + \lambda^2 x_1 + d - \frac{P_{12}}{P_{22}} e_2). \tag{3.26}$$

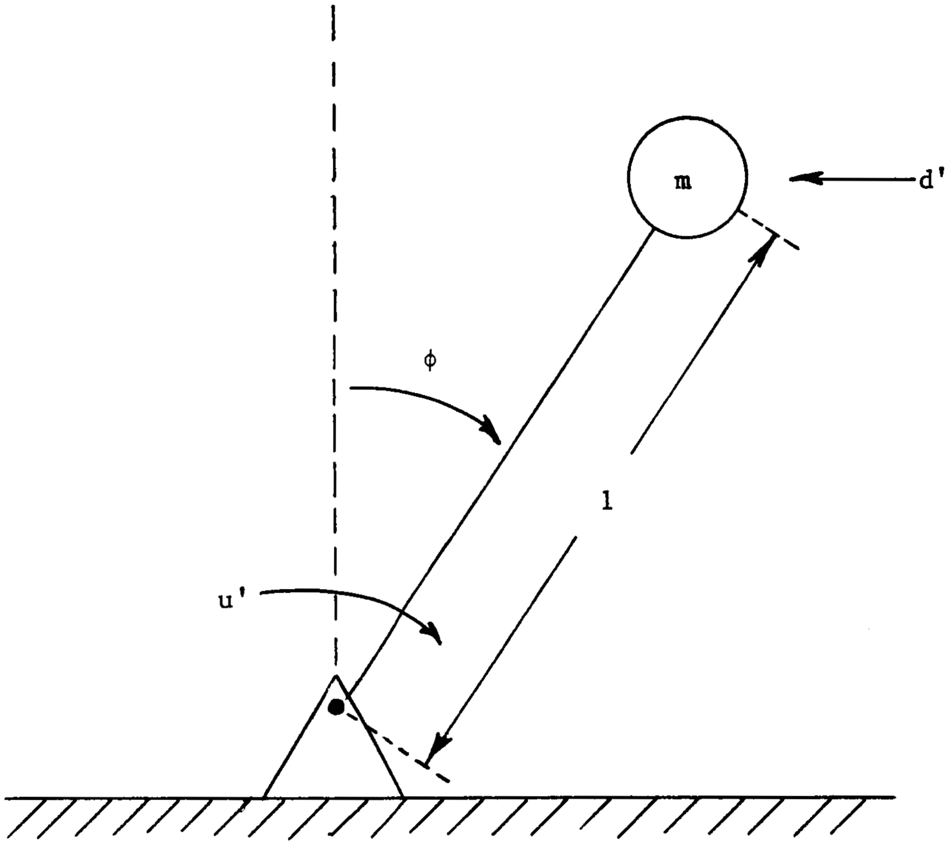

Figure 3.3 SECOND ORDER EXAMPLE

The motion within the strip $|\gamma| \leq \delta_T$ is given by

$$\dot{e}_1 = -\frac{P_{12}}{P_{22}} e_1 + \frac{\beta}{P_{22}} + d \tag{3.27}$$

and

$$e_2 = -\frac{P_{12}}{P_{22}} e_1 + \frac{\beta}{P_{22}} \; . \tag{3.28}$$

The reachable set of equation 3.27 is as derived by Higdon[8] is simply

$$e_1 \leq (\frac{1}{P_{12}})\delta_T + (\frac{P_{22}}{P_{12}})d_{max} \tag{3.29}$$

where

$$d_{max} = \mathrm{MAX}(|d|) \tag{3.30}$$

and therefore

$$e_2 \leq \frac{2\delta_T}{P_{22}} + d_{max}. \tag{3.31}$$

Equations 3.29 and 3.31 describe the ultimate error bound.

The maximum value of $\frac{P_{12}}{P_{22}} e_2$, E, can be determined from equation 3.31 as

$$E = \frac{2P_{12}}{P_{22}} \delta_T + \frac{P_{12}}{P_{22}} d_{max}. \tag{3.32}$$

Noting that $H = d_{max}$, the region R_w is defined by

$$|x_1| < \frac{1}{\lambda^2}(U_0 - \quad M \quad - (\frac{2P_{12}}{P_{22}^2} \delta_T + (\frac{P_{12}+P_{22}}{P_{22}})d_{max}). \tag{3.33}$$

The model chosen is of the form suggested previously, illustrated in Figure 3.1. The bound on the input reference was chosen such that the motion of the model within its reachable set always caused the plant to remain within R_w.

The region on controllability as discussed by Higdon is given by

$$|\lambda^2 x_1 + \lambda x_2| \leq U_0. \tag{3.34}$$

This region, which will be called R*, represents that region in $\underline{x}$ space outside of which no control law can force the system motion to be bounded.

For the purpose of illustration the following values were assumed for the example.

MODEL: $a_1 = 1$

$a_2 = 2$

$\dot{s}_n = 0.3$

PLANT: $\lambda^2 = 8.55$

$U_0 = 1$

CONTROLLER:

$\delta_R = 0.05$

$\delta_N = 0$

$d_{max} = 0.05$

$P_{12} = 1$

$P_{22} = 1$

The regions R_w, R'_w, and R* for this system are shown in Figure 3.4. The system was simulated on an analog computer and it was found that no disturbance less than d_{max}, and no input into the model could cause instability of the system. Error convergence was also investigated and a typical trajectory is shown in Figure 3.5.

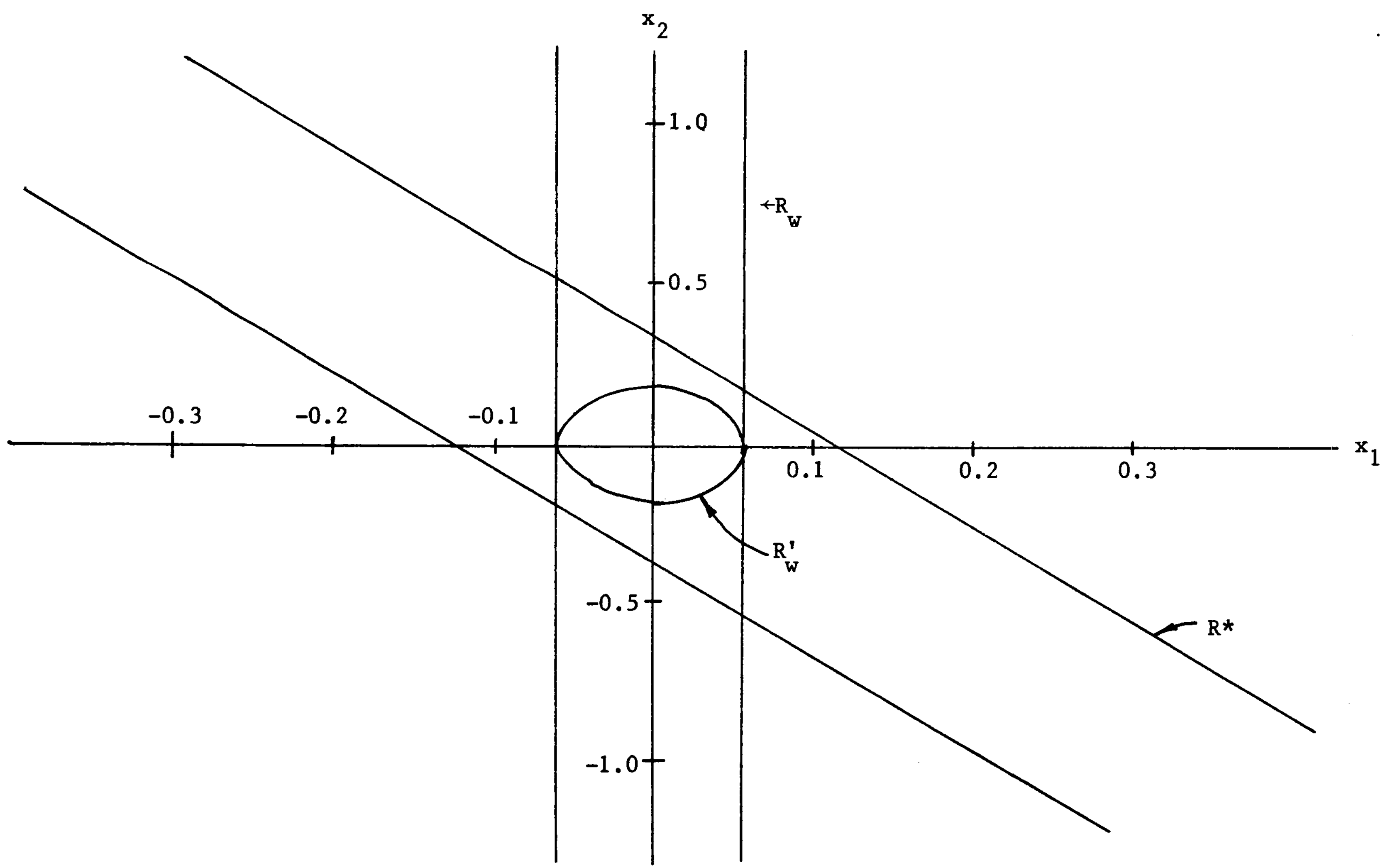

Figure 3.4 SECOND ORDER EXAMPLE - PHASE PLANE

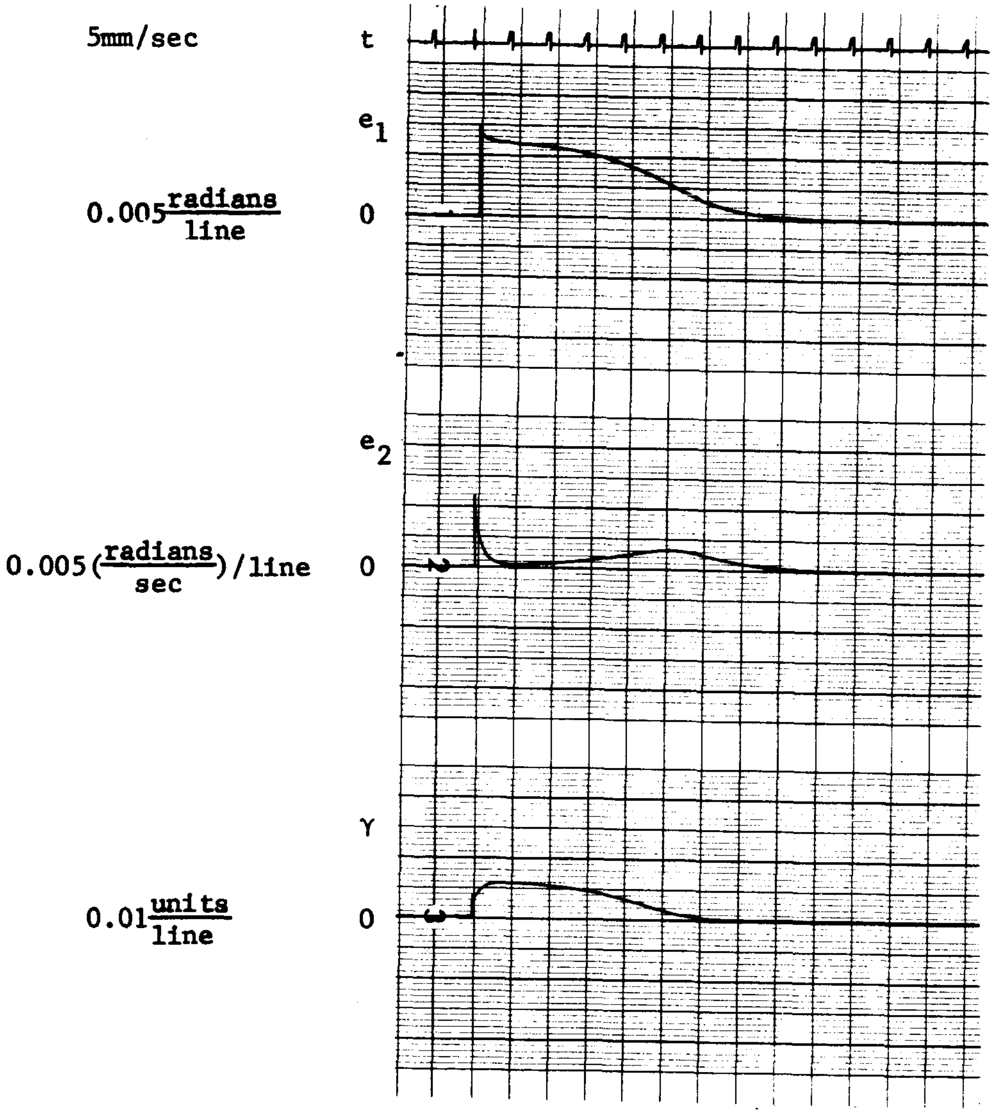

Figure 3.5 SECOND ORDER EXAMPLE - ERROR RESPONSE

CHAPTER IV

THE REGULATOR PROBLEM

Introduction

One interesting simplification of the synthesis technique occurs when the model is eliminated. It was the investigation of this problem that led to many of the more general results in this thesis. For this problem, the region R becomes a function only of $\underline{P}$, $\underline{x}$ and $\underline{h}$, thus the definition of R_w requires no assumptions about the system motion being within the error bound.

Statement of the Problem

The system is assumed to be the same as that considered in Chapter II as described by equation 2.1,

$$\dot{\underline{y}} = A\underline{y} + \underline{f}u + \underline{g}(\underline{y},\underline{z},t). \tag{2.1}$$

The objective is to define a controller which will guarantee stability in some region of the state space, while returning the system to the vicinity of $\underline{y} = 0$.

The approach to be taken is the same as in Chapter II, and the equations developed in Chapters II and III can be applied directly to this system by noting that with the model eliminated

$$\underline{s} = \underline{0}, \tag{4.1}$$

$$\dot{s}_n = 0, \tag{4.2}$$

and

$$\underline{e} = -\underline{x}. \tag{4.3}$$

Application of the Synthesis Technique

Thus the ideal controller is given by

$$u = U_0 \mathrm{SGN}(-P_{1n}x_1 - P_{2n}x_2 - \dots - P_{nn}x_n) \tag{4.4}$$

and the region R in which this control law assures $\dot{v} < 0$ is defined by

$$\begin{aligned}\mathrm{SGN}(\gamma) = \mathrm{SGN}(U_0 \mathrm{SGN}(\gamma) - a_1x_1 + (\frac{P_{1n}}{P_{nn}} - a_2)x_2 + \\ + (\frac{P_{2n}}{P_{nn}} - a_3)x_3 + \dots + (\frac{P_{n-1n}}{P_{nn}} - a_n)x_n \\ + \frac{P_{1n}}{P_{nn}} h_1 + \frac{P_{2n}}{P_{nn}} h_2 + \dots + \frac{P_{n-1n}}{P_{nn}} h_{n-1} + h_n)\end{aligned} \tag{4.5}$$

The region R_w can be defined by substituting the maximum value of the terms involving $\underline{h}$, thus R_w is defined by

$$(U_0-H) \geq \left|-a_1x_1 + (\frac{P_{1n}}{P_{nn}} - a_2)x_2 + (\frac{P_{2n}}{P_{nn}} - a_3)x_3 + \dots + (\frac{P_{n-1n}}{P_{nn}} - a_n)x_n\right|. \tag{4.6}$$

The requirement on $\underline{P}$ for bounded stable motion within the strip $|\gamma| < \delta_T$ is the same as in Chapter II, that is, the matrix C must be a stability matrix, since motion within the strip is defined by

$$\underline{\dot{x}}' = C\underline{x}' - \underline{\alpha}. \tag{4.7}$$

and

$$x_n = -\frac{P_{1n}}{P_{nn}}x_1 - \frac{P_{2n}}{P_{nn}}x_2 - \dots - \frac{P_{n-1n}}{P_{nn}}x_n - \frac{\beta}{P_{nn}}. \tag{4.8}$$

The ultimate bound on the motion in $\underline{x}$ as t approaches infinity is given by the reachable set of equation 4.7.

It is instructive at this point to consider the second order example of Chapter III.

Second Order Example

The control law for the second order example of the previous Chapter becomes

$$u = U_0 \mathrm{SGN}(-P_{12}x_1 - P_{22}x_2), \tag{4.9}$$

and the region R_w for this controller is defined by

$$(U_0 - H) \geq \left|\lambda^2 x_1 + \frac{P_{12}}{P_{22}} x_2\right|. \tag{4.10}$$

The region R_w and the region R_w' which was determined experimentally for this system are illustrated along with the region R* in Figure 4.1, where the system parameters are those of the previous Chapter, except that $P_{12} = 6.5$ and $P_{22} = 1.0$.

The actual region from which the system could be returned to origin was found experimentally and this region is illustrated in Figure 4.2.

An interesting special case is when the switching line is parallel to the boundary of R*, i.e., when $P_{12} = \lambda$ and $P_{22} = 1$. For this case the region R_w is maximized and corresponds exactly to the region R_w'. Furthermore, the actual region for which stability could be assured as determined experimentally is also identical to R_w. The regions R_w and R* for this system are illustrated in Figure 4.3.

It should be noted that it is not usually practical to choose $\underline{P}$ such that R_w is maximized, since the $\underline{P}$ also determines the nature of the motion along the switching line.

The Ideal Regulator Problem

The investigation of the ideal regulator problem leads to some significant conclusions about the chatter motion near $\gamma = 0$. In fact,

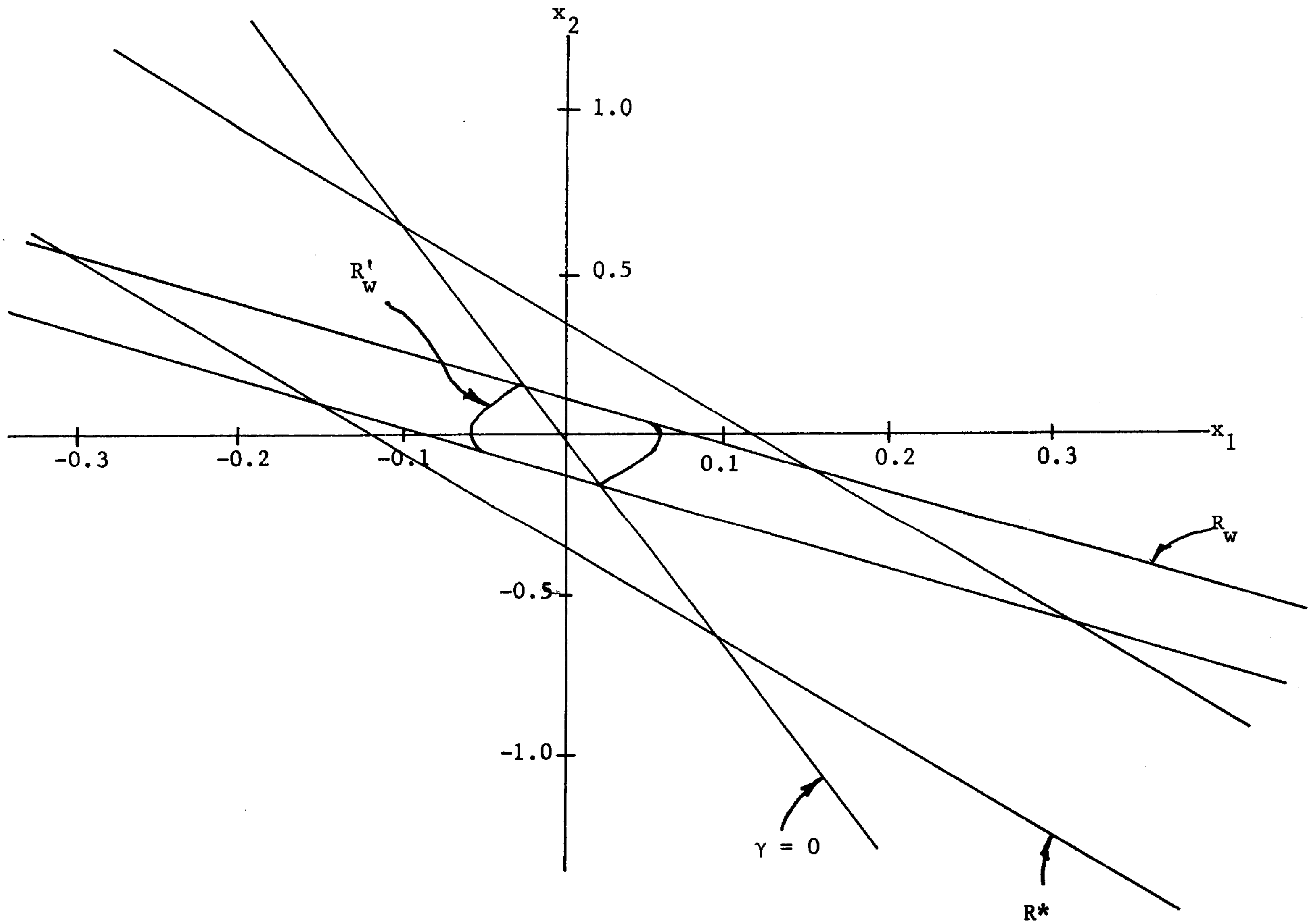

Figure 4.1 REGULATOR PROBLEM - PHASE PLANE

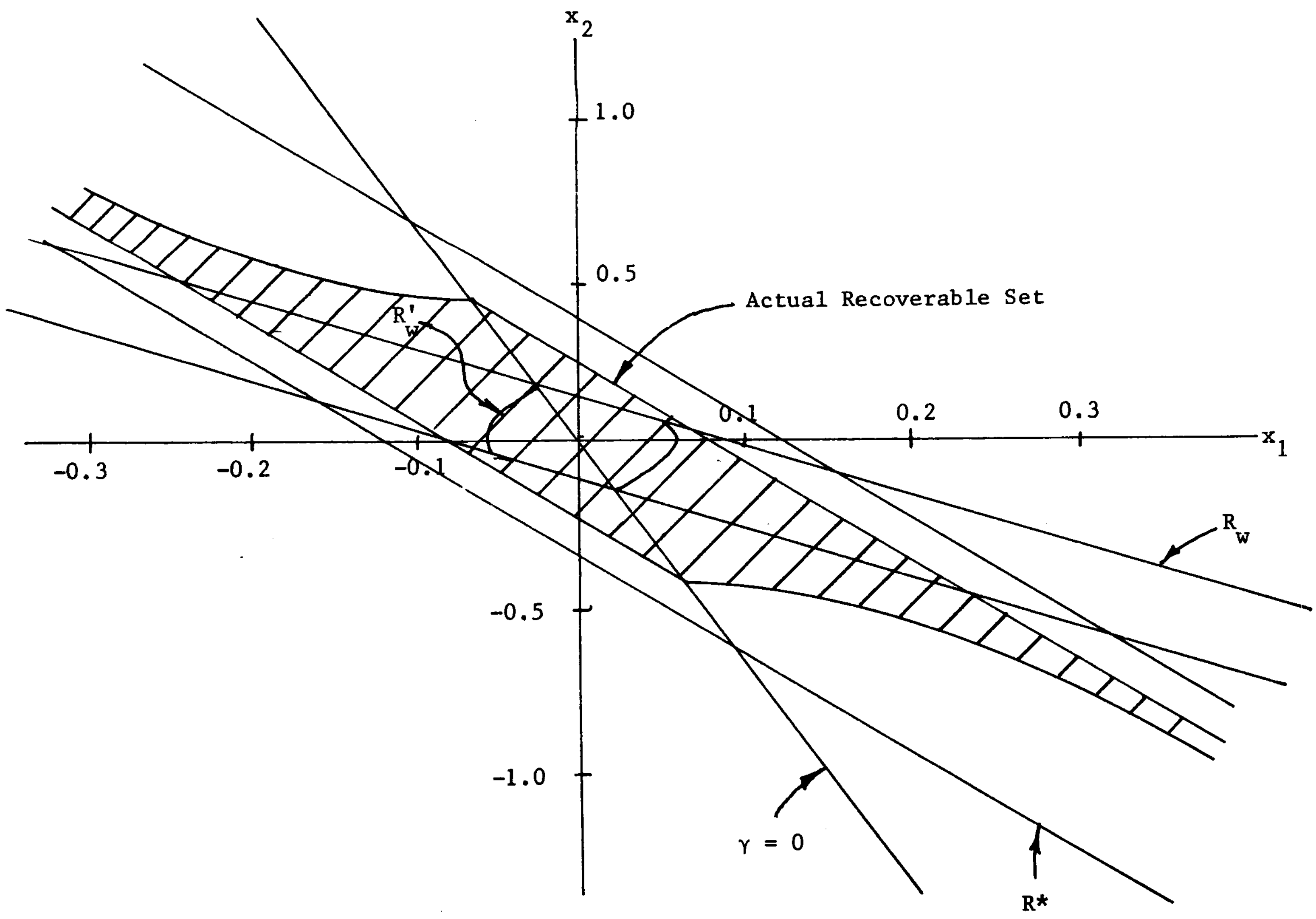

Figure 4.2 REGULATOR PROBLEM - ACTUAL RECOVERABLE SET

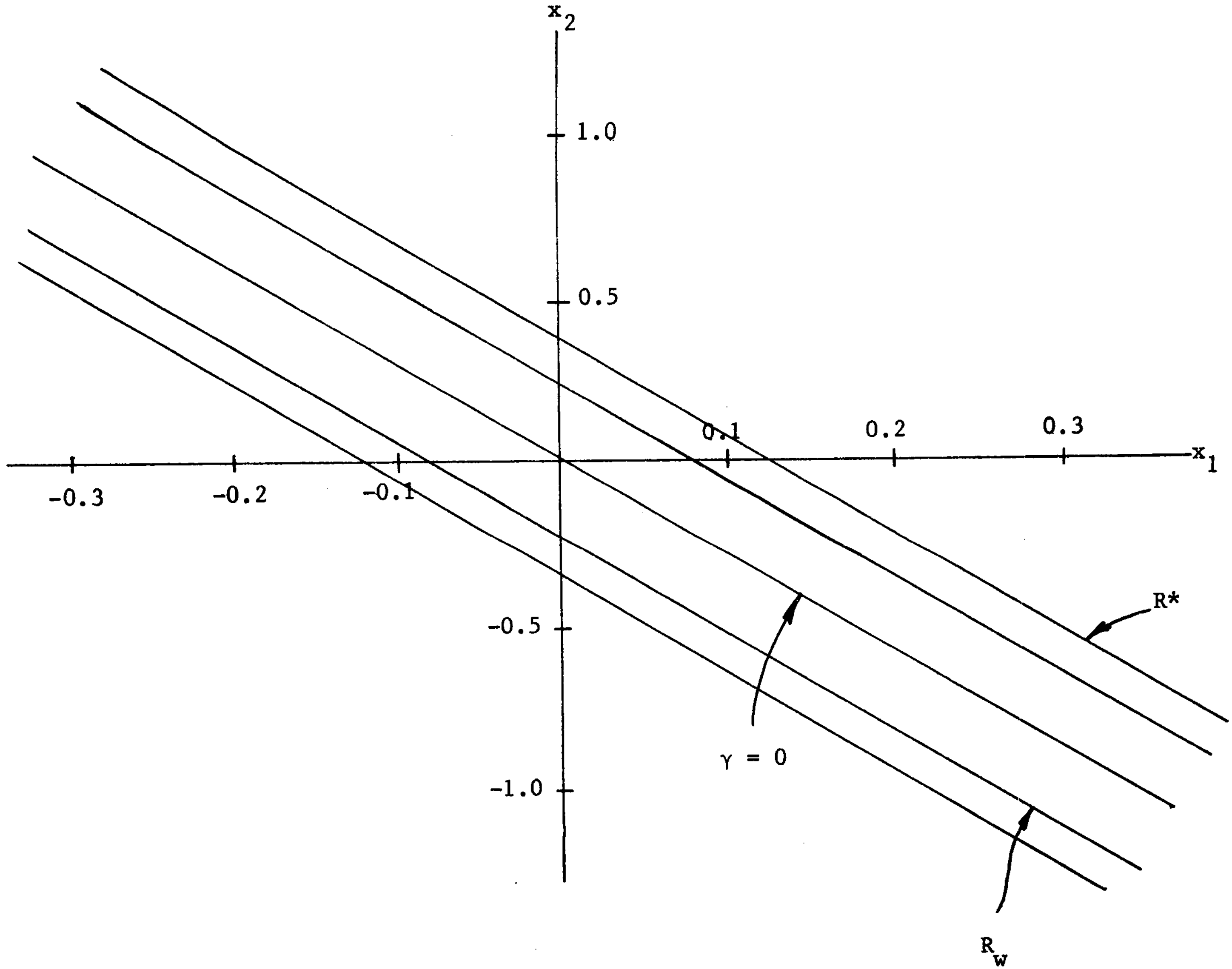

Figure 4.3 REGULATOR PROBLEM - MAXIMUM R_w

application of the synthesis technique readily yields the section of a switching line about which chatter motion can exist.

The ideal problem to be discussed is given by letting

$$\underline{h} = \underline{0} \tag{4.11}$$

and

$$\delta_T = 0 \tag{4.12}$$

in the system of equation 2.1. In this case, it is possible to treat the region R, defined by

$$\mathrm{SGN}(\gamma) = \mathrm{SGN}(U_0 \mathrm{SGN}(\gamma) - a_1 x_1 + (\frac{P_{1n}}{P_{nn}} + a_2)x_2 + \ldots$$

$$+ (\frac{P_{n-1n}}{P_{nn}} + a_n)x_n), \tag{4.13}$$

directly, since it is a function of $\underline{x}$ alone.

Within R the control law will force the system to move toward the switching line $\gamma = 0$ when $\gamma \neq 0$. Thus, the conditions for chatter motion, as outlined by Higdon, are satisfied on those parts of the switching line which lie in the interior of the region R, therefore, it can be concluded that the system will exhibit chatter about the segment of the line $\gamma = 0$ which lies inside R.

CHAPTER V

A SIXTH ORDER EXAMPLE

Introduction

The problem originally undertaken by the author was that of balancing an upright flexible beam mounted on a frictionless cart by the application of a control force acting horizontally on the cart, Figure 5.1. This problem is of some significance since it is analogous to the problem encountered in the attitude control of a flexible missle during lift-off. In as much as problems encountered in the application of the Liapunov synthesis technique to this system provided the incentive for the theoretical investigations contained in this thesis, it is only fitting therefore to treat this problem in the conclusion of this work.

The approach usually taken in problems of this type is to discretize the beam into N segments. The equal-length segments are connected in a chain-like fashion, with the connection of the segments consisting of a spring hinge representing the elastic stiffness of the beam at that particular station, Figure 5.2. The linearized equations of motion of such a beam mounted on a frictionless cart have been derived by Schaefer for an arbitrary number of segments. Theoretically any desired accuracy can be achieved by choosing N sufficiently large.

It is assumed that it is only desired to actively control cart position and the first bending mode. The technique used by Schaefer[10] to reduce the 2N + 2 order system, resulting from the equations of motion, to the desired sixth order system is to transform the system

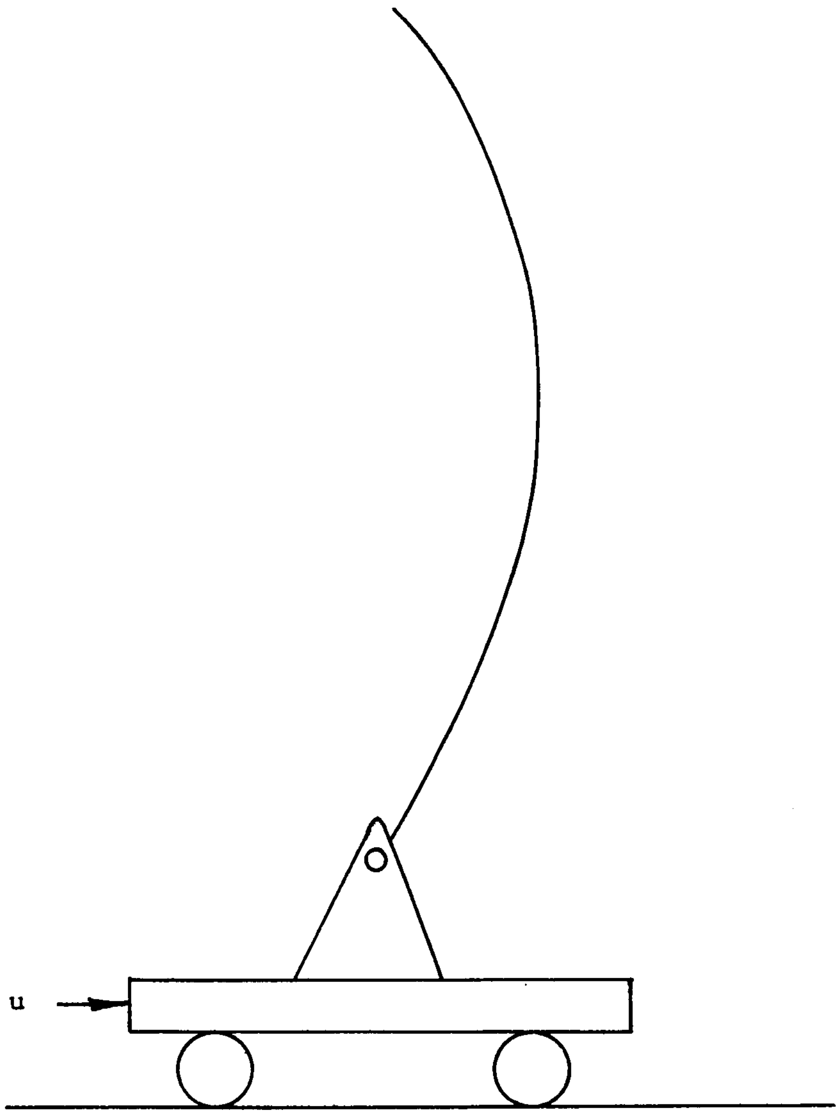

Figure 5.1 THE CONTINUOUS FORM OF THE MECHANICAL MODEL

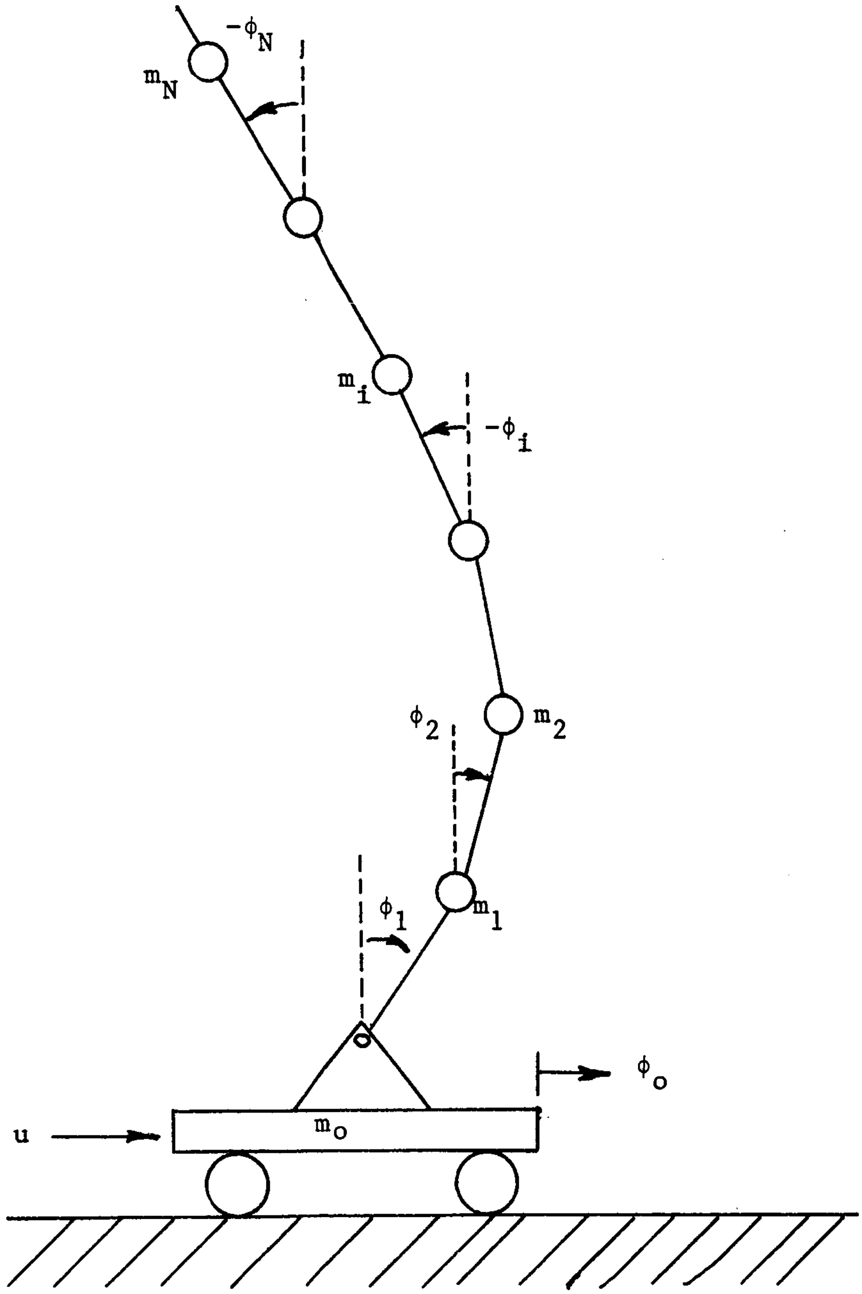

Figure 5.2 THE DISCRETE MODEL OF THE FLEXIBLE BEAM

equations to Jordan normal form, and discard the variables representing the higher order bending modes. This approach is desirable when one wants to accurately represent a specific mechanical model. However, for the purpose of this example, which is to demonstrate the application of the synthesis technique to a particular class of problems and not to represent an accurate description of any specific mechanical model, it is sufficient to treat the discrete two-segment beam problem, Figure 5.3. The solution to this problem is analogous to the continuous beam problem. The effect of disturbance and noise are not considered in this example, since the evaluation of the error bound requires the investigation of the reachable set of a fifth-order system with a complex input function. The existing techniques of Higdon[8] and Lemay[9] cannot handle this problem realistically.

Equations of Motion

The two pendulum segments, each of length h, are assumed to be connected by a spring hinge of stiffness EI. The two point masses corresponding to each segment have equal mass m, and the angles of the segments with respect to the vertical are designated as ϕ_1 and ϕ_2 as indicated in Figure 5.3. The cart is of mass m_0, and the total mass of the system is referred to as M_T,

$$M_T = m_0 + 2m. \tag{5.1}$$

The control force, u, is assumed to act horizontally on the cart.

The linearized equations of motion for this system as derived by Schaefer are given by

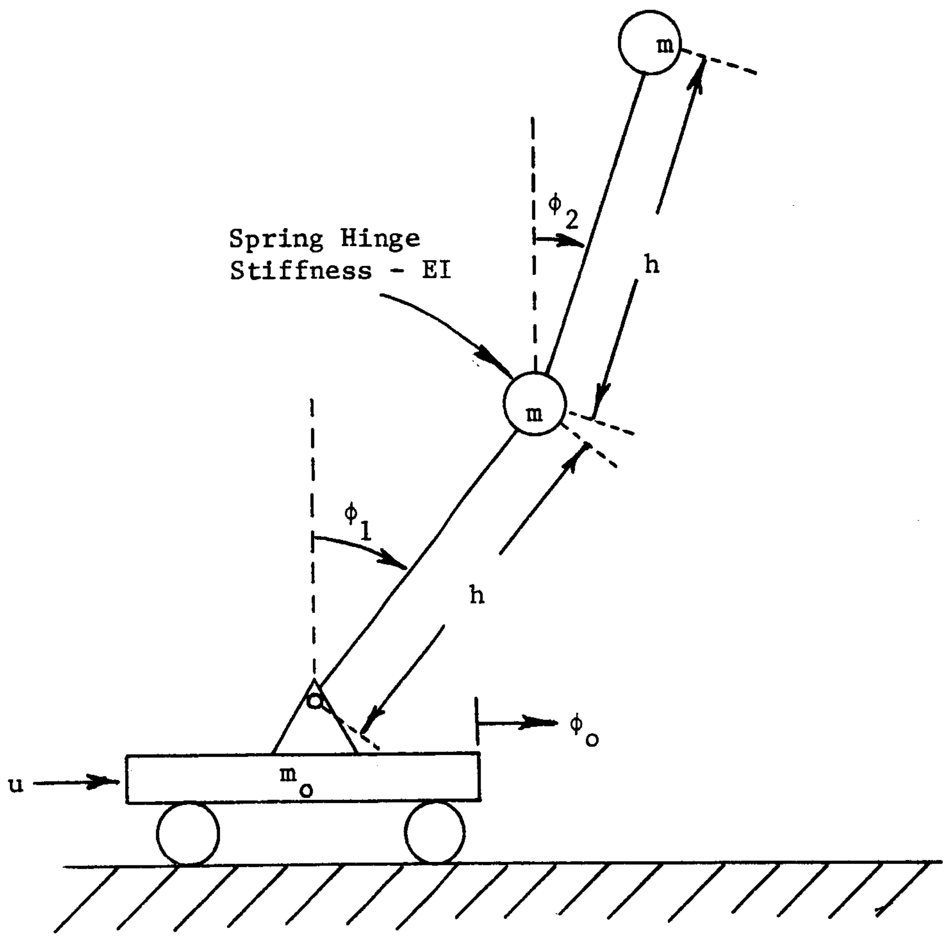

Figure 5.3 SIXTH ORDER EXAMPLE

$$\begin{bmatrix} \frac{M_T}{hm} & 2 & 1 \\ 2 & 2h & h \\ 1 & h & h \end{bmatrix} \ddot{\underline{\phi}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & (2\mu^2-2g) & -2\mu^2 \\ 0 & -2\mu^2 & (2\mu^2-g) \end{bmatrix} \underline{\phi} = \begin{bmatrix} \frac{1}{hm} \\ 0 \\ 0 \end{bmatrix} u, \tag{5.2}$$

where

$$\underline{\phi} = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} \tag{5.3}$$

and

$$\mu^2 = \frac{EI}{2h^2 m} \tag{5.4}$$

The basic assumption made in the linearization of the system equations is that the angles ϕ_1 and ϕ_2 are sufficiently small such that the acceleration forces due to $\ddot{\phi}_1$ and $\ddot{\phi}_2$ can be treated as acting horizontally.

Multiplying through by the inverse of the coefficient matrix of $\ddot{\underline{\phi}}$ gives the equations in the somewhat more convenient form

$$\ddot{\underline{\phi}} = \begin{bmatrix} 0 & -K_1 & -K_2 \\ 0 & -K_3 & -K_4 \\ 0 & -K_5 & -K_6 \end{bmatrix} \underline{\phi} + \begin{bmatrix} \frac{1}{m_0} \\ -\frac{1}{hm_0} \\ 0 \end{bmatrix} u \tag{5.5}$$

where

$$K_1 = -\left(\frac{EI-2mgh^2}{h^2 M_o}\right) \tag{5.6}$$

$$K_2 = \frac{EI}{h^2 M_0}, \tag{5.7}$$

$$K_3 = \left(\frac{2EI-2mgh^2}{h^3m}\right) + \left(\frac{EI-2mgh^2}{h^3M_0}\right), \tag{5.7}$$

$$K_4 = -\left(\frac{2EI-mgh^2}{h^3m} + \frac{EI}{h^3M_0}\right), \tag{5.9}$$

$$K_5 = -\left(\frac{3EI-2mgh^2}{h^3m}\right), \tag{5.10}$$

and

$$K_6 = \left(\frac{3EI-2mgh^2}{h^3m}\right). \tag{5.11}$$

Defining a new vector $\underline{y}$ as

$$\underline{y} = \begin{bmatrix} \phi_0 \\ \dot{\phi}_0 \\ \phi_1 \\ \dot{\phi}_1 \\ \phi_2 \\ \dot{\phi}_2 \end{bmatrix} \tag{5.12}$$

allows equation 5.5 to be written as the first order matrix equation,

$$\ddot{\underline{y}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -K_1 & 0 & -K_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -K_3 & 0 & -K_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -K_5 & 0 & -K_6 & 0 \end{bmatrix} \underline{y} + \begin{bmatrix} 0 \\ 1/M_0 \\ 0 \\ -1/hM_0 \\ 0 \\ 0 \end{bmatrix} u. \tag{5.13}$$

Tranformation to Canonic (Phase-Variable) Form

It is desired to define a transformation

$$\underline{y} = K\underline{x} \tag{5.14}$$

which transforms equation 5.13 to the canonic form

$$\dot{\underline{y}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u. \tag{5.15}$$

The transformation matrix as computed by the equations of Rane is

$$K = \begin{bmatrix} \frac{K_5 g}{hm_0} & 0 & \frac{\alpha}{hm_0} & 0 & \frac{1}{m_0} & 0 \\ 0 & \frac{K_5 g}{hm_0} & 0 & \frac{\alpha}{hm_0} & 0 & \frac{1}{m_0} \\ 0 & 0 & \frac{K_5}{hm_0} & 0 & -\frac{1}{hm_0} & 0 \\ 0 & 0 & 0 & \frac{K_5}{hm_0} & 0 & -\frac{1}{hm_0} \\ 0 & 0 & \frac{K_5}{hm_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{K_5}{hm_0} & 0 & 0 \end{bmatrix} \tag{5.16}$$

where

$$\alpha = \frac{SEI - 4mgh^2}{h^3 m} \tag{5.17}$$

Its inverse K^{-1} is given by

$$K^{-1} = \begin{bmatrix} \frac{hm_0}{K_5 g} & 0 & \frac{h^2 m_0}{K_5 g} & 0 & -(\frac{h^2 m_0}{K_5 g}(K_5+\alpha)) & 0 \\ 0 & \frac{hm_0}{K_5 g} & 0 & \frac{h^2 m_0}{K_5 g} & 0 & -(\frac{h^2 m_0}{K_5 g}(K_5-\alpha)) \\ 0 & 0 & 0 & 0 & \frac{hm_0}{K_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{hm_0}{K_5} \\ 0 & 0 & -hm_0 & 0 & hm_0 & 0 \\ 0 & 0 & & -hm_0 & 0 & hm_0 \end{bmatrix}. \qquad (5.18)$$

The resulting canonic equation is given by equation 5.15 where

$$a_1 = a_2 = a_4 = a_6 = 0, \qquad (5.19)$$

$$a_3 = -K_5(K_3 + K_4) \qquad (5.20)$$

$$a_5 = (K_3 - K_5) \qquad (5.21)$$

The application of the synthesis technique for this ideal system with no disturbances or noise, simply involves the selection of a model to give a "desired" response, and the determination of a switching line that will satisfy the conditions for stability.

The region R in which $\dot{v} < 0$ for this system is given by

$$SGN(\gamma) = SGN(U_0 SGN(\gamma) - \dot{s}_6 - a_3 x_3 - a_5 x_5 + \frac{P_{16}}{P_{66}} e_2 + \frac{P_{26}}{P_{66}} e_3 + \frac{P_{36}}{P_{66}} e_4 + \frac{P_{46}}{P_{66}} e_5 + \frac{P_{56}}{P_{66}} e_6). \qquad (5.22)$$

The switching function $SGN(\gamma)$ was implemented with δ_R essentially zero, thus the error bound was sufficiently small such that the error terms could be neglected in the definition of R_w. With this assumption the region R_w is given by

$$U_0 - \quad M \quad > |-a_3x_3 - a_5x_5|. \tag{5.23}$$

In terms of the original variables the region R_w is given by

$$U_0 - \quad M \quad > |(K_3 - K_5)\phi_1 + (K_4 + K_5)\phi_2|. \tag{5.24}$$

The evaluation of the controllable zone could not be carried out due to the complexity of the system. Equation 5.24 indicates however that the zone is determined mainly in terms of the permissable angles ϕ_1 and ϕ_2.

Selection of the Model

Since it was not the purpose of this example to meet any specific performance requirements the model was chosen with simple linear feedback into a saturation function. The feedback coefficients were chosen by selecting a transfer function in the linear region as

$$\frac{s_1}{s_{1_{REF}}} = \frac{1}{(\tau s+1)^6}\ . \tag{5.25}$$

The form of the model is illustrated in Figure 5.4.

Selection of the Switching Line

In the ideal system, the only necessary requirement on the switching $\gamma = 0$, where

$$\gamma = P_{16}x_1 + P_{26}x_2 + P_{36}x_3 + P_{46}x_4 + P_{56}x_5 + P_{66}x_6, \quad (5.26$$

is that the equation

$$P_{66}\lambda^5 + P_{56}\lambda^4 + P_{46}\lambda^3 + P_{36}\lambda^2 + P_{26}\lambda + P_{16} = 0 \tag{5.27}$$

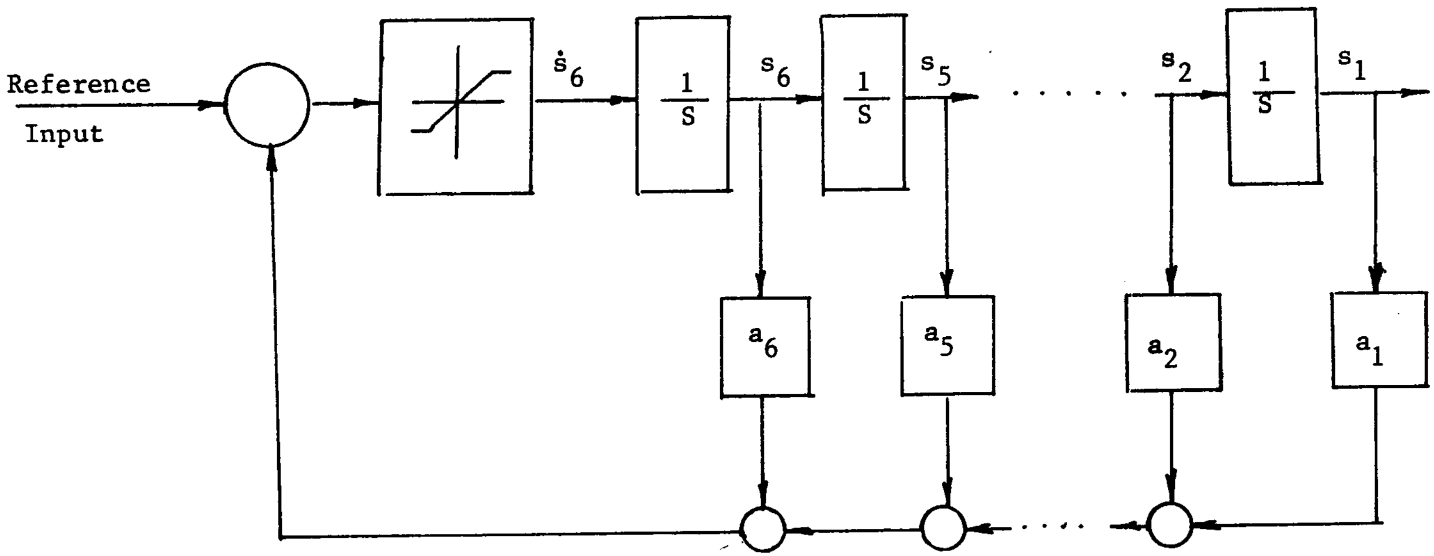

Figure 5.4 SIXTH ORDER MODEL

be Hurwitz. This requirement can be readily achieved by simply choosing the coefficients in γ to be the coefficients of an equation known to be stable. An equation of the form

$$(s + a)^5 = 0 \tag{5.28}$$

was chosen for this example.

Simulation Results

The sixth-order system was simulated on an analog computer, with parameters chosen as

$EI = 25$ lb/ft^2

$m = 0.05$ slugs

$m_o = 0.10$ slugs

$h = 2$ ft

$g = 32.2$ ft/sec^2

$u = 0.01$ lb

$M = 0.003$

The linear response of the model was chosen to be characterized by the characteristic equation

$$(s + 1)^6 = 0. \tag{5.29}$$

The switching line was chosen such that the characteristic equation of the C matrix could be written as

$$(s + 1)^5 = 0, \tag{5.30}$$

thus the switching line was given by

$$e_1 + 5e_2 + 10e_3 + 10e_4 + 5e_5 + e_6 = 0. \tag{5.31}$$

A typical step response is shown in Figure 5.5. The quantity $\phi_1 - \phi_2$ represents a measure of the bending mode vibrations.

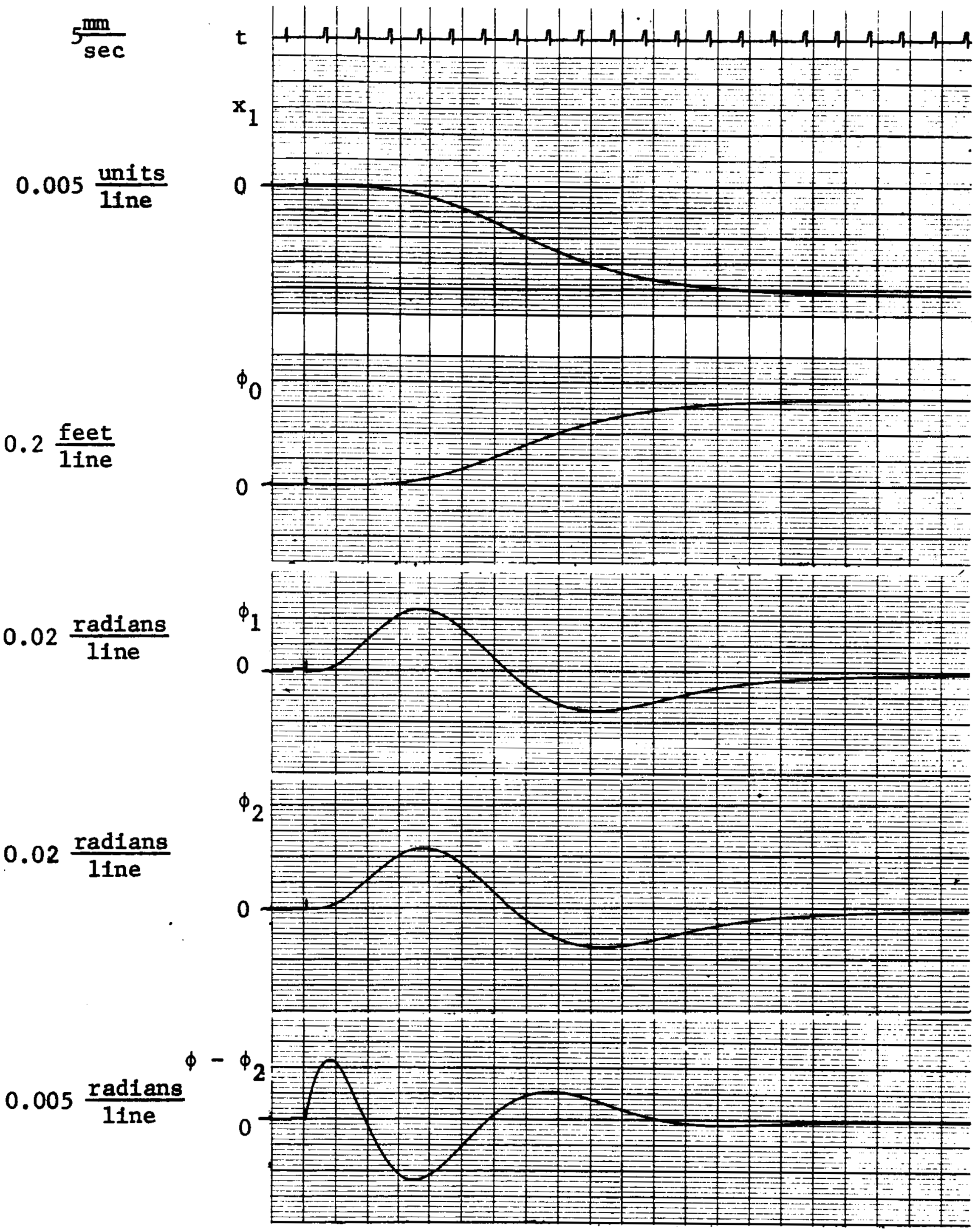

Figure 5.5 SIXTH ORDER SYSTEM - STEP RESPONSE

Error convergence was examined, and a typical system response for initial conditions such that the error was initially outside the error bound, and a typical trajectory is shown in Figure 5.5. The error response along the line $\gamma = 0$ should agree with that predicted by equation 3.18,

$$\dot{\underline{e}}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -5 & -10 & -10 & -5 \end{bmatrix} \underline{e}' \qquad (3.18)$$

or

$$(s + 1)^5 e_1 = 0. \qquad (5.32)$$

It is noted in Figure 5.6 that the response of e_1 for the period when $\gamma = 0$ is consistent with equation 5.32.

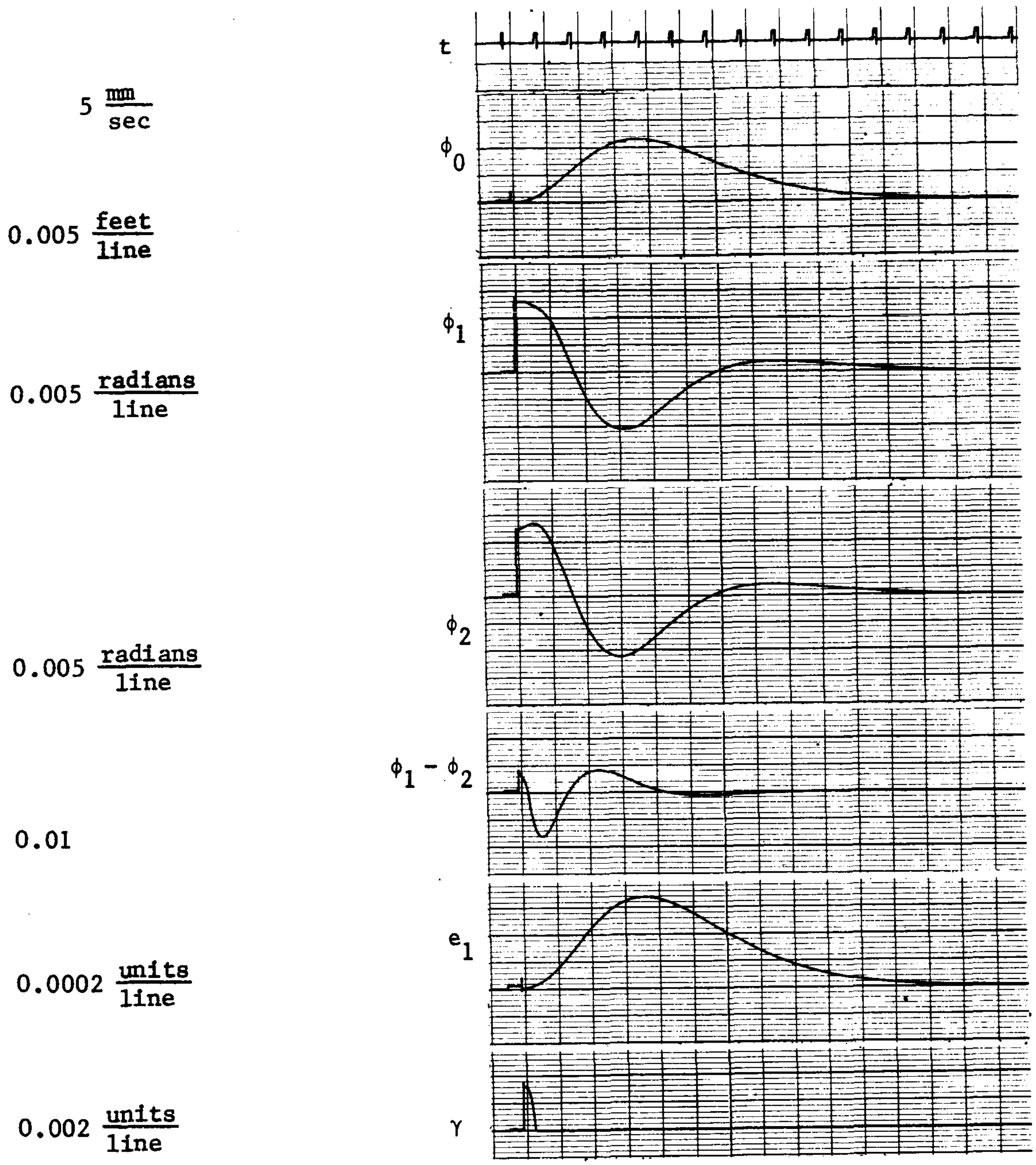

Figure 5.6 SIXTH ORDER SYSTEM - RESPONSE TO AN INITIAL ERROR

CHAPTER VI

AREAS FOR FUTURE INVESTIGATION

Many of the problems encountered in this thesis indicate a need for future research. The fact that the reachable sets of systems with complex input functions cannot be realistically evaluated by existing techniques surely indicates a need for future research in this area.

The necessity of the transformation to canonic form precluded the treatment of the parameter variation problem, since the transformation is a function of the system parameters. The possibility of handling the equations in the non-canonic form could possibly allow the treatment of the parameter variation problem.

No analytical expression could be derived for the controllable zone of the system, and at present this region can only be found by experimental techniques. The development of a technique by which some indication of the controllable zone could be determined would represent a significant contribution to the usefulness of the technique in practical systems.

It was pointed out that for a given system design, there exists a permissable set of inputs to the model. The method of determining this set of inputs for a given system has not been studied in any detail. The possibility of designing a model whose permissable set of inputs encompassed all possible inputs provides incentive in this area.

The use of the semi-definite Liapunov function which was zero on the switching line raises the possibility of defining more complex

Liapunov functions which are zero on non-linear switching lines. The advantages of a non-linear switching line would be a larger controllable zone. The first step in this area would be a treatment of the piece-wise linear switching line.

The treatment of the regulator problem of Chapter IV indicates the possibility of designing a system without the use of a model. Obviously, there would be a limited set of inputs which could be applied to the system but this is the case in a system with a model.

One final area of future investigation, could be the generalization of the example of Chapter V to the flexible beam problem, and the construction of an actual mechanical model. The model would be quite useful in evaluating the results of investigations suggested in this Chapter.

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